Interactions and non-interactions of gravity waves with the mean flow

We have seen how eddies can affect the mean flow, through eddy flux divergences and energy conversions. Despite the presence of such terms in the equations, however, it turns out that under surprisingly general conditions the eddies do not alter the mean flow. There are several related theorems that demonstrate this “non-interaction” of the eddies with the mean flow. They are called, reasonably enough, “non-interaction theorems.” The earliest such ideas were published by Eliassen and Palm (1961), and the following discussion of this section is based on their paper. The same material is also discussed in more detail, and in somewhat more general form, in Chapter 8 of Lindzen’s (1990) book.

Consider the equation of zonal motion in the simplified form

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial x}.
\]

(1)

We have omitted rotation, sphericity, friction, and meridional motions. Eq. (1) can apply, for example, to small-scale gravity waves forced by flow over topography. Let

\[
\begin{align*}
    u &= U + u', \quad U = U(z), \\
    w &= w', \\
    p &= \bar{p} + p', \quad \bar{p} = \bar{p}(z), \\
    \rho &= \bar{\rho} + \rho', \quad \bar{\rho} = \bar{\rho}(z).
\end{align*}
\]

(2)

We interpret the primed quantities as small-amplitude wave-like perturbations with zero means. Recall that

\[
\rho \frac{\partial u}{\partial t} \sim -\frac{\partial}{\partial z} \left( \rho \omega' u' \right).
\]

(3)
We are interested in what determines the wave momentum flux divergence, \( \frac{\partial}{\partial z} (\rho w'u') \).

Substitute (2) into (1) and linearize, to obtain

\[
\bar{\rho} \frac{\partial u'}{\partial t} = -\left( \bar{\rho} U \frac{\partial u'}{\partial x} + \bar{\rho} w' \frac{\partial U}{\partial z} + \frac{\partial p'}{\partial x} \right).
\]

(4)

Assume that the perturbations are steady, so that

\[
\frac{\partial u'}{\partial t} = 0.
\]

(5)

This implies both that the waves are neutral, i.e., neither amplifying or decaying, and also that they are stationary, i.e., their phase speed is zero. The latter assumption is reasonable, e.g., for mountain waves. Then (4) reduces to

\[
0 = \bar{\rho} U \frac{\partial u'}{\partial x} + \bar{\rho} w' \frac{\partial U}{\partial z} + \frac{\partial p'}{\partial x} = \frac{\partial}{\partial x} \left( \bar{\rho} U u' + p' \right) + \bar{\rho} w' \frac{\partial U}{\partial z}.
\]

(6)

This is the form of the steady-state equation of motion that we will use.

Next, multiply (6) by \((\bar{\rho} U u' + p')\), to obtain

\[
0 = \frac{\partial}{\partial x} \left[ \frac{(\bar{\rho} U u' + p')^2}{2} \right] + \bar{\rho}^2 U \frac{\partial U}{\partial z} w'u' + \bar{\rho} w' \frac{\partial U}{\partial z} p'.
\]

(7)

The term involving \(\frac{\partial}{\partial x}\) vanishes when integrated over the whole domain, leaving

\[
\frac{\partial U}{\partial z} \left( U \int_{-\infty}^{\infty} \bar{\rho} w'u' \,dx + \int_{-\infty}^{\infty} w'p' \,dx \right) = 0,
\]

(8)

which can be simplified to

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provided that \( \frac{\partial U}{\partial z} \neq 0 \).

Eq. (9) is an important result. It shows that the wave momentum flux, \( \int_{-\infty}^{\infty} \bar{\rho}w'u' \, dx \), and the wave energy flux, \( \int_{-\infty}^{\infty} w'p' \, dx \), are closely related. At a “critical” level, where \( U = 0 \), the wave energy flux must vanish; the only other possibility is that our assumptions, e.g., a steady state with no friction, do not apply at the critical level. For a wave forced by flow over a mountain, the energy flux is, of course, upward, but (9) shows that it goes to zero at a critical level. This means that the wave does not exist above the critical level. The upward propagation of the wave is blocked at the critical level.

Eq. (9) also shows that a wave with an upward energy flux will produce a downward momentum flux in westerlies and an upward momentum flux in easterlies. In either case, the wave is driving the mean flow towards zero, i.e., it is exerting a drag on the mean flow.

Let \( e_E \) be the total eddy energy per unit mass associated with the wave (the sum of the eddy kinetic, eddy internal, and eddy potential energies). It can be shown that \( e_E \) satisfies

\[
\frac{\partial}{\partial x} (\bar{\rho}e_E U + p'u') + \frac{\partial}{\partial z} (p'w') = -\bar{\rho}u'w' \frac{\partial U}{\partial z}.
\]

(10)

The right-hand side of (10) is a “gradient production” term that represents conversion of the kinetic energy of the mean state into the total eddy energy, \( e_E \). Eq. (10) simply says that the production term on the right-hand side is balanced by the transport terms on the left-hand side. Integration over the domain gives

\[
\frac{\partial}{\partial z} \int_{-\infty}^{\infty} p'w' \, dx = -\frac{\partial U}{\partial z} \int_{-\infty}^{\infty} \bar{\rho}u'w' \, dx.
\]

(11)

This means that the wave energy flux divergence balances conversion to or from the kinetic energy of the mean flow.

By combining (9) and (11) we can show that
Therefore, when $U \neq 0$, the wave momentum flux $\int_{-\infty}^{\infty} \hat{\rho}u'w' \, dx$ is independent of height. This is very important because, as shown by (3), it implies that the wave momentum flux has no effect on $U(z)$, except at the critical level where $U = 0$. The wave momentum flux is absorbed at the critical level. From (3), it follows that $U$ will tend to change with time at the critical level, so $U$ will become different from zero. Therefore, the critical level will move.

If we allowed the phase speed $c$ to be non-zero, we would find $U - c$ everywhere in place of $U$. The momentum would be absorbed at the critical level where $U = c$.

Since (12) tells us that $\int_{-\infty}^{\infty} \hat{\rho}u'w' \, dx$ is independent of height (where $U \neq 0$), we see from (9) that the wave energy flux is just proportional to $U$. Alternatively, we can combine (9) and (12) to write

$$\frac{1}{U} \int_{-\infty}^{\infty} w'p' \, dx = \text{constant}. \quad (13)$$

The conserved quantity $\frac{1}{U} \int_{-\infty}^{\infty} w'p' \, dx$ is called the “wave action.” Eq. (9) can be written as “wave action plus wave momentum flux = zero.”

Since the mid-1980s, there has been a lot of interest in the effects of gravity wave momentum fluxes on the general circulation; because the waves act to decelerate the mean flow, these interactions are referred to as “gravity wave drag” (McFarlane, 1987). Most of the discussion so far has been on gravity waves forced by flow over topography, although recently gravity waves forced by convective storms are receiving a lot of attention (e.g., Fovell et al., 1992).

Fig. 9.1 shows the deceleration of the zonally averaged zonal wind induced by gravity-wave drag in a general circulation model, as reported by McFarlane (1987). Here the gravity wave drag has been parameterized using methods that we will not discuss, which are based on the assumption that the waves are produced by flow over mountains. The plot shows the “tendency” of the zonally averaged zonal wind due to this orographic gravity-wave drag, for northern-winter conditions. The actual response of the zonally averaged zonal wind is shown in Fig. 9.2. The changes are very large. In order for thermal wind balance to be maintained, there must be corresponding changes in the zonally averaged temperature; these are shown in Fig. 9.3.
The polar troposphere has warmed dramatically, to be consistent with the weaker westerly jet. The changes shown in Fig. 9.2 and Fig. 9.3 make the model results more realistic than before, suggesting that gravity-wave drag is an important process in nature.

**Vertical and meridional propagation of planetary waves**

The following discussion is based on the famous paper by Charney and Drazin (1961), which deals with the vertically propagating planetary waves. Closely related work can be found in Dickinson (1968a) and Matsuno (1970).

Let \( T_s(p) \) be a basic-state temperature profile, and define \( \alpha_s, \theta_s, \) and \( \rho_s \) accordingly. The quasi-geostrophic form of the potential vorticity equation is

\[
\frac{\partial}{\partial t} + \mathbf{V}_g \cdot \nabla p \right) q = 0,
\]

where

\[
(14)
\]
\[ q = f + \zeta + \frac{\partial}{\partial p} \left( \frac{f_0}{S_p} \frac{\partial \phi}{\partial p} \right) \] (15)

is the quasi-geostrophic pseudo-potential vorticity, \( S_p \equiv -\frac{\alpha_s}{\theta_s} \frac{\partial \theta_s}{\partial p} \) is the static stability, and in the last term of (15) \( f \) has been replaced by \( f_0 \). [See Chapter 8 of Holton (1992).] We are working on a \( \beta \)-plane, such that \( f = f_0 + \beta y \). Note that \( q \) is essentially determined by the absolute vorticity and the change of temperature with height, and that (14) does not contain a vertical advection term.

Exercise: Show that

\[ \frac{\partial[q]}{\partial t} = -\frac{\partial}{\partial y} \left[ v_x^* q^* \right]. \] (16)
As we will see later, the expression on the right-hand side of (17) is the divergence of the Eliassen-Palm flux. Eq. (17) expresses a very important relationship. It says that the meridional eddy flux of potential vorticity is related to the convergence of the meridional eddy flux of zonal momentum, and to the rate of change with height of the meridional eddy sensible heat flux.

When we form the convergence of the eddy potential vorticity flux, i.e., \(-\frac{\partial [v_s q^*]}{\partial y}\), (17) will give us \(\frac{\partial}{\partial y}\left\{ -\frac{\partial [u_s v_s^*]}{\partial y} \right\}\). This affects the meridional shear of \([u]\). We will also get a term proportional

\[
\begin{bmatrix}
v_s^* q^*
v_s^* \zeta_s^*
\end{bmatrix} = \begin{bmatrix} v_s^* \zeta_s^* \end{bmatrix} - \frac{\partial}{\partial p} \left( \frac{Rf_0}{p S_p} \begin{bmatrix} v_s^* \Gamma^* \end{bmatrix} \right)
\]

\[
= -\frac{\partial}{\partial y} \begin{bmatrix} u_s^* v_s^* \end{bmatrix} - \frac{\partial}{\partial p} \left( \frac{Rf_0}{p S_p} \begin{bmatrix} v_s^* \Gamma^* \end{bmatrix} \right).
\]
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This affects the static stability.

We adopt the “log pressure” coordinate

\[
 z(p) \equiv -\left(\frac{RT_0}{g}\right) \ln\left(\frac{p}{p_0}\right),
\]

(18)

where \( T_0 \) is a constant reference temperature. With the use of (18), (15) can be rewritten as

\[
 q = f + \nabla^2 \psi + \frac{1}{\rho_s} \frac{\partial}{\partial z} \left( \rho_s \frac{f_0^2}{N^2} \frac{\partial \psi}{\partial z} \right),
\]

(19)

where

\[
 \psi \equiv \frac{\phi}{f_0}
\]

(20)

is called the “geostrophic stream function,” and the Brunt-Vaisala frequency \( N \), satisfies

\[
 N^2 \equiv \frac{g}{\theta_s} \frac{\partial \theta_s}{\partial z}.
\]

(21)

Recall that

\[
 v_g = \frac{\partial \psi}{\partial x} \quad \text{and} \quad u_g = -\frac{\partial \psi}{\partial y}.
\]

(22)

Linearizing (14) about the zonal-mean state gives

\[
 \left( \frac{\partial}{\partial t} + \left[ u \right] \frac{\partial}{\partial x} \right) q^* + v_g \frac{\partial [q]}{\partial y} = 0.
\]

(23)

We look for solutions of the form

\[
 \psi^* = \text{Re}\left\{ \hat{\psi}(y, z) e^{i(x-ct)} \right\},
\]

(24)
\[ q^* = \text{Re}\{\hat{q}(y,z)e^{ik(x-ct)}\}. \]  

(25)

Substitution of (19), (24), and (25) into (23) gives

\[(|u| - c)\hat{q} + \hat{\psi} \frac{\partial[q]}{\partial y} = 0, \]

(26)

where

\[ \hat{q} = -k^2\hat{\psi} + \frac{\partial^2 \hat{\psi}}{\partial y^2} + \frac{1}{\rho_s} \frac{\partial}{\partial z} \left( \rho_s \frac{f_0^2}{N^2} \frac{\partial \hat{\psi}}{\partial z} \right). \]

(27)

Using (27), we can rewrite (26) as

\[ \frac{\partial^2 \hat{\psi}}{\partial y^2} + \frac{1}{\rho_s} \frac{\partial}{\partial z} \left( \rho_s \frac{f_0^2}{N^2} \frac{\partial \hat{\psi}}{\partial z} \right) = -\left( \frac{1}{|u| - c} \frac{\partial[q]}{\partial y} - k^2 \right) \hat{\psi}. \]

(28)

This is a fairly general form of the quasi-geostrophic wave equation that we want to analyze, but we will simplify it considerably before doing so.

As wave energy propagates up to higher levels, it encounters decreasing values of \( \rho_s \). The energy-density (energy per unit volume) scales like \( \rho_s (k\psi)^2 \), so if the energy density is constant with height, \( \hat{\psi} \) must increase like \( \frac{1}{\sqrt{\rho_s}} \). Because of this effect, the equations become simpler if we introduce a scaled value of \( \hat{\psi} \):

\[ \psi \equiv \frac{\sqrt{\rho_s}}{N} \hat{\psi}. \]

(29)

Note that here \( \psi \) (no hat) is the scaled value; the meaning of \( \psi \) now departs from that used in (20). We also note that
\[
\frac{1}{\rho_S} \frac{\partial}{\partial z} \left( \frac{\rho_S f_0^2}{N^2} \frac{\partial \hat{\psi}}{\partial z} \right) = \frac{f_0^2}{\rho_S} \frac{\partial}{\partial z} \left( \sqrt{\frac{\rho_S}{N}} \frac{\partial}{\partial z} \left( \sqrt{\frac{\rho_S}{N}} \psi \right) - \sqrt{\frac{\rho_S}{N}} \frac{\partial}{\partial z} \left( \sqrt{\frac{\rho_S}{N}} \psi \right) \right)
\]
\[
= \frac{f_0^2}{\rho_S} \left( \frac{\partial}{\partial z} \left( \sqrt{\frac{\rho_S}{N}} \right) \frac{\partial}{\partial z} \left( \sqrt{\frac{\rho_S}{N}} \psi \right) + \sqrt{\frac{\rho_S}{N}} \frac{\partial^2}{\partial z^2} \left( \sqrt{\frac{\rho_S}{N}} \psi \right) \right)
\]
\[
- \frac{\partial}{\partial z} \left( \sqrt{\frac{\rho_S}{N}} \right) \frac{\partial}{\partial z} \left( \sqrt{\frac{\rho_S}{N}} \psi \right) - \sqrt{\frac{\rho_S}{N}} \frac{\partial^2}{\partial z^2} \left( \sqrt{\frac{\rho_S}{N}} \psi \right) \right) \text{.}
\]

(30)

Substituting from (29) and (30), we can rewrite (28) as

\[
\frac{\partial^2 \psi}{\partial y^2} + \frac{f_0^2}{N^2} \frac{\partial^2 \psi}{\partial z^2} = - \left( \frac{f_0^2}{N^2} \frac{n^2}{4H_0^2} \right) \psi \text{.}
\]

(31)

where

\[
n^2 \equiv \frac{4N^2H_0^2}{f_0^2} \left\{ \frac{1}{g} \frac{\partial}{\partial y} \left[ q \right] - k^2 - \frac{f_0^2}{\rho_S N} \frac{\partial^2}{\partial z^2} \left( \sqrt{\frac{\rho_S}{N}} \psi \right) \right\}
\]

(32)

is called the “index of refraction.” Here \( H_0 = \frac{RT_0}{g} \), where \( T_0 \) is a reference temperature. Eq. (32) is a form of the quasi-geostrophic wave equation. When \( n^2 > 0 \), \( \psi \) is oscillatory (propagating), and when \( n^2 < 0 \), \( \psi \) is “evanescent” (exponentially decaying away from the source of excitation).

Comparing (31) - (32) with (28), we see that the left-hand side of (31) has become simpler, but the expression for the index of refraction has become more complicated. Using some idealizations, we can simplify (32) drastically without altering its basic meaning. First, consider the special case of an isothermal atmosphere with \( T_s(p) \equiv T_0 = \text{constant} \). This is not unrealistic for the lower stratosphere. For an isothermal atmosphere \( N = \frac{g^2}{c_p T_0} = \text{constant} \) and \( \rho_s \sim e^{-\frac{z}{H_0}} \), so that (32) reduces to
\[ n^2 \equiv \frac{4N^2H_0^2}{f_0^2} \left( \frac{1}{|u|} \frac{\partial [q]}{\partial y} - k^2 \right) - 1. \]

(33)

Inspection of (33) shows that \([u] - c > 0\) is necessary for \(n^2 > 0\), i.e., for propagation.

Now we simplify further by concentrating on stationary waves, for which the phase speed, \(c\), is zero. This type of wave can be forced by flow over mountains, for example, as discussed in Chapter 8. Then (31) and (33) become

\[ \frac{\partial^2 \psi}{\partial y^2} + \frac{f_0^2}{N^2} \frac{\partial^2 \psi}{\partial z^2} = -\left( \frac{f_0^2}{N^2} \frac{n^2}{4H_0^2} \right) \psi, \]

(34)

\[ n^2 = \frac{4N^2H_0^2}{f_0^2} \left( \frac{1}{|u|} \frac{\partial [q]}{\partial y} - k^2 \right) - 1. \]

(35)

To simplify \(n^2\) even further, note from (19) that

Figure 9.4: The square of the index of refraction for summer and winter, averaged between 30º and 60ºN, for waves of different wavelengths, \(L\). The short-dashed lines correspond to \(L = 6,000 \text{ km}\), the long-dashed lines correspond to \(L = 10,000 \text{ km}\), and the solid lines correspond to \(L = 14,000 \text{ km}\). From Charney and Drazin (1961).
\[ \frac{\partial [q]}{\partial y} = \beta - \frac{\partial^2 [u]}{\partial y^2} - \frac{1}{\rho_s} \frac{\partial}{\partial z} \left( \frac{f_0^2}{N^2} \frac{\partial [u]}{\partial z} \right), \]

(36)

where \( \beta \equiv \frac{df}{dy} \). When the meridional and vertical shears of \([u]\) are not too strong,

\[ \frac{\partial [q]}{\partial y} \cong \beta \geq 0. \]

(37)

Using (37) in (35), we finally obtain

\[ n^2 \cong \frac{4N^2H_0^2}{f_0^2} \left( \frac{\beta}{[u]} - k^2 \right) - 1. \]

(38)

From (38), we see the following:

- To have propagation \((n^2 > 0)\), we need \( \beta/[u] > 0 \). Because \( \beta > 0 \), \([u]\) must be positive (westerly). Stationary Rossby waves cannot exist in easterlies, simply because they propagate westward relative to the air, so that easterlies cannot hold them in place. Recall that the summer hemisphere stratosphere is dominated by easterlies, while the winter hemisphere stratosphere is dominated by westerlies. Note, however, that large positive \([u]\) also makes \(n^2 < 0\). Stationary waves cannot propagate through very strong westerlies, which would sweep them downstream. Fig. 9.4, from Charney and Drazin (1961), shows the vertical distribution of \(n^2\) for summer and winter, averaged over the Northern Hemisphere middle latitudes, for stationary waves with three different wavelengths.

- Even when \( \beta/[u] > 0 \), for a given \([u]\), waves with large \(k\) (i.e., sufficiently short wavelength) cannot propagate. Short waves are, therefore, “trapped” near their excitation levels. Since \([u]\) has a maximum near the tropopause in middle latitudes, many short waves are trapped in the troposphere, even in winter. Only longer waves can propagate to great heights. This suggests that long waves will dominate in the stratosphere and mesosphere even more than they do in the troposphere.

- A level where \([u] = 0\) is called a “critical level” for stationary waves. Eq. (38) shows that, at a critical level, \(n^2 \to \infty\). Suppose that \([u] > 0\) below a critical level, and \([u] < 0\) above. Then, for waves excited at the lower boundary (e.g., by flow over topography),
upward propagation is completely blocked at the critical level, and no wave activity will be seen above the critical level.

Figure 9.5: These Northern Hemisphere data were collected during the International Geophysical Year. Geopotential heights for July 15, 1958 are shown on the left, and those for January 15, 1959 are shown on the right. The levels plotted are 500 mb, 100 mb, and 10 mb. From Charney (1973).

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Fig. 9.5 provides evidence that the theory is correct. It shows the geopotential height fields at 500 mb, 100 mb, and 10 mb, for Northern Hemisphere summer and winter. In winter, planetary waves clearly propagate upward to the 10 mb level, while in summer they do not. Note that the apparent horizontal scale of the dominant eddies increases upward, in winter. This is consistent with the theory, which predicts that the shorter modes are trapped at lower levels while

Figure 9.6: a) An idealized basic state zonal wind distribution (in m s\(^{-1}\)) for the Northern Hemisphere winter. b) The latitudinal gradient of the potential vorticity, \( \partial[q]/\partial \phi \), expressed as a multiple of the Earth's rotation rate. c) The refractive index square \( n^2 \), for the \( k = 0 \) wave. d) Computed distribution of energy flow in the meridional plane associated with zonal wave number 1. From Matsuno (1970).
longer modes can continue to propagate upward to great heights.

Waves can also be trapped at critical latitudes where \([u] = 0\). We could therefore define critical surfaces in the y-z plane.

If we allowed \(c \neq 0\), we would find that the critical surfaces are those for which \([u] - c = 0\).

Matsuno (1970) used for the Northern Hemisphere winter to compute \(\frac{\partial q}{\partial \phi}\), the index of refraction, and the energy flow in the latitude-height plane for zonal wave number 1. His results are shown in Fig. 9.6. The upward-propagating waves are directed equatorward by the variations of the index of refraction.
Fig. 9.7 shows that the eddy kinetic energy in the stratosphere is supplied by the troposphere. $KZ$ is converted into $AZ$, i.e., the meridional temperature gradient is increased by an indirect circulation. The general circulation of the lower stratosphere thus acts like a refrigerator.

**Sudden warmings**

The preceding discussion is highly relevant to stratospheric sudden warmings. This is how stratospheric sudden warmings work: The polar stratosphere is observed to warm up, while the lower latitudes cool off, as shown in the upper panel of Fig. 9.8. There is no significant change of the area-averaged temperature, however. This suggests that the sudden warming is due to a poleward redistribution of sensible heat. The polar westerlies are observed to weaken, and in some cases they give way to easterlies. There are two mechanisms that can change the zonally averaged temperature by meridional redistribution of sensible heat. The possibilities are:
• *A mean meridional circulation.* Sinking near the poles can produce adiabatic warming, while the compensating rising motion in lower latitudes gives adiabatic cooling. This MMC would be direct, at least to start with.

• *Poleward eddy energy transport.* In this case, eddies produce warming near the pole and cooling in lower latitudes. An MMC would be produced by the “apparent” heating and cooling due to the eddies. This MMC would feature rising at the pole to counteract the eddy warming there, and sinking in lower latitudes to counteract the eddy cooling there. This means that the MMC would be indirect.

In the first scenario, the westerlies will increase aloft. (Why?) Since the westerly shear must decrease during a sudden warming (Why?), there must be an even stronger intensification of the westerlies below. In the second scenario, on the other hand, the westerlies will weaken above. The observed transition to easterlies in the stratosphere supports the second hypothesis. We conclude that stratospheric sudden warmings are produced by poleward eddy heat fluxes.

Sudden warmings are characterized by intense wave activity, of planetary scale. The longitudinal phase is stationary, suggesting orographic forcing. Sudden warmings are relatively infrequent and weak in the Southern Hemisphere, where there is little orography.

There is a strong flux of wave energy from the troposphere to the stratosphere during winter. From our previous analysis, we know that such a wave will transport energy poleward. A sudden warming is triggered by the rapid growth of a quasi-stationary planetary wave in the troposphere, and its subsequent upward propagation into the stratosphere. As shown in Fig. 9.8 and Fig. 9.9, there is observational evidence for such wave growth, often in association with the formation of a blocking high (discussed later).

As it propagates upward, the planetary wave transports energy poleward, warming the pole and consequently weakening the westerlies aloft. In extreme cases, the westerlies are actually changed to easterlies. When easterlies form above, the wave propagation is blocked, so the wave energy is concentrated near the critical level. This leads to really sudden warming.
Matsuno (1970, 1971; see also Matsuno and Nakamura, 1979) gave a physical explanation of sudden warmings and was the first to successfully simulate sudden warmings with a numerical model.

Figure 9.8: a) Latitude-time section of zonal mean temperature (K) measured by channel A for 31 December 1970 to 16 January 1971. Regions of temperature lower than 258 K are shaded. A major warming had a peak at this level on 9 January at 80°N. b) Latitude-time section of amplitude of zonal wavenumber one of channel A temperature (K) for 31 December 1970 to 16 January 1971. Maximum amplitude occurred on 4 January at 65°N. From Barnett (1974).

Matsuno (1970, 1971; see also Matsuno and Nakamura, 1979) gave a physical explanation of sudden warmings and was the first to successfully simulate sudden warmings with a numerical model.
Now we investigate under what conditions planetary waves can transport energy and momentum. The quasi-geostrophic form of the thermodynamic energy equation is (e.g., Holton, 1992)

\[
\frac{\partial}{\partial t} \left( \mathbf{V}_g \cdot \nabla \right) \frac{\partial \phi}{\partial p} + S_p \omega = 0.
\]

(39)

Here \( \mathbf{V}_g \) is the geostrophic wind, which is important for the following discussion, and \( \nabla \) is \( \nabla_p \). Eq. (39) can be written as

Figure 9.9: 10 mb charts during a sudden warming in January 1963. Height contours (solid lines) at 32 dm intervals. Isotherms (dashed lines) at 10 K intervals. Note that the process spans 9 days. From Sawyer (1965).

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where \( w \) is defined by \(-\omega / (\rho_s g)\). Here \( \psi_z = \partial \psi / \partial z \), and \( z \) is the “log-p” coordinate defined by (18). Linearizing (40) gives

\[
\left( \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} \right) \psi_z^* - v \frac{\partial}{\partial z} [u] + \frac{N^2}{f_0} w^* = 0 .
\]

(41)

Here we have used the thermal wind equation. Multiplying (41) by \( \psi_z^* \), we obtain a form of the “temperature variance equation:"

\[
\left( \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} \right) \left( \frac{1}{2} (\psi_z^*)^2 \right) - v \psi_z^* \frac{\partial [u]}{\partial z} + \frac{N^2}{f_0} w^* \psi_z^* = 0 .
\]

(42)

Note the two gradient production terms.

Take the zonal mean of (42), so that the \( [u] \frac{\partial}{\partial x} \) term drops out. Rearrange to isolate the meridional energy flux by itself on the left-hand side:

\[
\left[ v \psi_z^* \right] \frac{\partial [u]}{\partial z} = \frac{\partial}{\partial t} \left[ \frac{1}{2} (\psi_z^*)^2 \right] + N^2 \left[ \frac{w^* \psi_z^*}{f_0} \right] .
\]

(43)

Note that \( [w^* \psi_z^*] / f_0 > 0 \) implies an upward temperature flux, in either hemisphere. Also \( [v \psi_z^*] > 0 \) implies a poleward temperature flux, in either hemisphere.

First, consider a baroclinically amplifying wave, for which \( \frac{\partial}{\partial t} \left[ (\psi_z^*)^2 \right] > 0 \) and the wave temperature flux is upward. From (43), we see that a baroclinically amplifying wave produces a poleward temperature flux (in either hemisphere) when \( \frac{\partial [u]}{\partial z} > 0 \), i.e., when the temperature is decreasing towards the pole. Such a temperature flux is downgradient, so the gradient-production term is positive.
Next, consider a neutral wave of the form $e^{i(kx-ct)}$, for which $\frac{\partial}{\partial t} = -c \frac{\partial}{\partial x}$, where $c$ is real. Multiply (41) by $\psi^*$ and take the zonal mean, to obtain

$$((u] - c)[v^*\psi^*] = N^2 \left[ \frac{w^*\psi^*}{f_0} \right].$$

(44)

Note that $[w^*\psi^*/f_0] > 0$ means an upward propagation of wave energy in either hemisphere. Recall also that $[u] - c > 0$ is needed in order for the wave to propagate. It follows that an upward-propagating neutral wave transports energy poleward. Such a wave might be forced, for example, by flow over mountains.

In summary, poleward energy transport is produced by either a baroclinically amplifying wave with $\frac{\partial[u]}{\partial z} > 0$ or a neutral wave that propagates upward.

Applying the eddy PV equation (23) to a neutral wave gives

$$((u] - c) \frac{\partial q_*}{\partial x} + v^* \frac{\partial [q]}{\partial y} = 0.$$  

(45)

Multiply (45) by $\psi^*$ and take the zonal mean to show that

$$[v^*q^*] = 0 \text{ except where } [u] = c,$$

(46)
i.e., except at a critical line. This very important result shows that neutral waves produce no potential vorticity flux except at a critical line. It follows from (16) that neutral waves do not affect $[q]$ except at a critical line. This is a non-interaction theorem for quasi-geostrophic planetary waves, analogous to the non-interaction theorem for gravity waves obtained by Eliassen and Palm (1961).

Referring back to (17), we see that $[v^*q^*] = 0$ means

$$-\frac{\partial}{\partial y}[u^*v^*] + \frac{f_0^2}{\rho_s} \frac{\partial}{\partial z} \left( \frac{\rho_s}{N^2} [v^*\psi^*] \right) = 0.$$  

(47)

As mentioned earlier, the expression equal to zero in (47) is the divergence of the Eliassen-Palm flux. Vertically integrate (47) through the depth of the atmosphere to obtain
\[-\int_{0}^{p_{0}} \frac{\partial}{\partial y} \left[ u^{*} v^{*} \right] dp = g \int_{0}^{\rho_{s}} \frac{\rho_{s}}{N^{2}} \left[ v^{*} \psi^{*} \right] dp \]

(48)

for the neutral waves. The left-hand side represents the vertically integrated convergence of meridional momentum flux, and the right-hand side represents the near-surface value of the eddy meridional energy flux. Recall from (44) that an upward-propagating neutral wave produces a poleward energy flux, i.e. \( [v^{*} \psi^{*}] > 0 \). It follows from (48) that

\[-\int_{0}^{p_{0}} \frac{\partial}{\partial y} \left[ u^{*} v^{*} \right] dp > 0 \text{ for an upward-propagating neutral wave.} \]

(49)

This means that the vertically integrated meridional momentum flux convergence tends to accelerate the vertically integrated \([u]\). In other words, the eddy momentum flux tries to increase the speed of the jet! This is consistent with the observation that \(KE \rightarrow KZ\), discussed in Chapter 7. On the other hand, because the waves also transport temperature poleward, they will tend to reduce the meridional temperature gradient and so (as implied by thermal-wind balance) tend to reduce the strength of the jet. The momentum flux and heat flux thus have opposing effects on the mean flow. If these two opposing effects cancel, then the eddies have no net effect on the mean flow.

An upward propagating neutral wave in westerly shear tends to produce a downward momentum flux at the Earth's surface. To see why, consider the zonal momentum equation, in Cartesian coordinates for simplicity:

\[ \frac{\partial u}{\partial t} + \frac{\partial}{\partial x} (uu) + \frac{\partial}{\partial y} (uv) + \frac{1}{\rho_{s}} \frac{\partial}{\partial z} (\rho_{s} uw) = -\frac{\partial \phi}{\partial x} + f v. \]

(50)

We have neglected the metric term and assumed no friction above the boundary layer. Taking the zonal mean of (50) gives

\[ \frac{\partial [u]}{\partial t} + \frac{\partial}{\partial y} [u^{*} v^{*}] + \frac{1}{\rho_{s}} \frac{\partial}{\partial z} (\rho_{s} [u^{*} w^{*}]) = f [v]. \]

(51)

Here advection of \([u]\) by \([v]\) and \([w]\) is neglected; this is justified in the midlatitude winter. To the extent that \([v]\) is geostrophic, it vanishes anyway. Now assume \( \frac{\partial [u]}{\partial t} = 0 \), consistent with \( \frac{\partial [q]}{\partial t} = 0 \). This leads to

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\[
\frac{\partial}{\partial y} [u^* v^*] = -\frac{1}{\rho_s} \frac{\partial}{\partial z} \left( \rho_s [u^* w^*] \right) + f [v].
\]

(52)

Integrating (52) vertically with respect to mass, using \( \int_0^\infty \rho_s [v] dz = 0 \), and using (49), we find that

\[
\int_0^\infty \frac{\partial}{\partial z} \left( \rho_s [u^* w^*] \right) dz > 0.
\]

(53)

We know that \( \rho_s [u^* w^*] \) must vanish at great height, so we conclude that

\[
\rho_s [u^* w^*] < 0.
\]

(54)

This shows that, near the lower boundary, friction and/or mountain torque must carry westerly momentum into the Earth's surface, in the presence of an upward propagating planetary wave. An alternative interpretation is that frictional and/or mountain torque, in a belt of westerlies where (54) is satisfied, will produce an upward-propagating planetary wave that transports energy poleward.

Compare (49) and (54). The meridional momentum flux accelerates the westerlies, while the vertical momentum flux decelerates them.

**Eliassen-Palm Theorem for more general balanced flows**

Previously we discussed non-interaction theorems for pure gravity waves and for quasi-geostrophic waves on a \( \beta \)-plane. It was discovered during the 1970's that non-interaction theorems can be derived for very general balanced flows. The following discussion provides an example. The discussion is based on Andrews et al. (1987).

The zonally averaged equations in spherical coordinates can be written as

\[
\frac{\partial [M]}{\partial t} + [v] \frac{\partial [M]}{\partial \phi} + [w] \frac{\partial [M]}{\partial z} - [F_s] a \cos \phi = \frac{-1}{a \cos \phi} \frac{\partial}{\partial \phi} \left( [M^* v^*] \cos \phi \right) \frac{\partial}{\partial \phi} \left( \rho_s [M^* w^*] \right),
\]

(55)

\[
= \frac{-1}{a \cos \phi} \frac{\partial}{\partial \phi} \left( [v^*]^2 \right) \cos \phi - \frac{1}{\rho_s} \frac{\partial}{\partial z} \left( \rho_s [v^* w^*] \right) - \frac{[v^*]^2 \tan \phi}{a},
\]

(56)
\[
\frac{\partial[\theta]}{\partial t} + \frac{[v]}{a} \frac{\partial[\theta]}{\partial \phi} + [w] \frac{\partial[\theta]}{\partial z} - [Q] = \frac{-1}{a \cos \phi} \frac{\partial}{\partial \phi} \left( [\theta^* v^*] \cos \phi \right) - \frac{1}{\rho_s} \frac{\partial}{\partial z} \left( \rho_s [w^* \theta^*] \right)
\]

(57)

\[
\frac{1}{a \cos \phi} \frac{\partial}{\partial \phi} \left( \rho_s [v] \cos \phi \right) + \frac{\partial}{\partial z} \left( \rho_s [w] \right) = 0,
\]

(58)

\[
\frac{\partial[\phi]}{\partial z} - \frac{\rho \sigma}{H} \frac{\zeta}{\eta} = 0
\]

(59)

Here \( z \equiv -H \log \left( \frac{p}{p_0} \right) \) is the vertical coordinate, and \( w \equiv \frac{Dz}{Dt} \). See the QuickStudy on vertical coordinate systems. The scale height \( H \) is \( \frac{RT_0}{g} \), where \( T_0 \) is a constant. In (57), \( Q \) represents a heating process. In the above equations, \( \rho_s (z) \equiv \rho_0 e^{-\frac{z}{H}} \), where \( \rho_0 \) is a constant. We have assumed for simplicity that the temperature is uniform with height, but this assumption is not really needed.

We assume that the meridional momentum equation, (56), can be approximated by gradient wind balance, i.e.

\[
[u] \left( f + \frac{[u] \tan \phi}{a} \right) + \frac{1}{a} \frac{\partial[\phi]}{\partial \phi} \equiv 0.
\]

(60)

The smallness of \([v]\) outside the tropics, discussed in Chapter 3, implies that the first three terms on the left-hand side of (56) are small. Eq. (60) can then be obtained by neglecting friction and the various eddy terms on the right-hand side of (56). Eq. (60) is essential to the following argument.

We define a “residual mean meridional circulation” \((0, V, W)\) by

\[
V \equiv [v] - \frac{1}{\rho_s} \frac{\partial}{\partial z} \left( \rho_s [v^* \theta^*] \right),
\]

(61)
\[
W \equiv [w] + \frac{1}{a \cos \phi} \frac{\partial}{\partial \phi} \left( \cos \phi \frac{[v^* \theta^*]}{\partial \theta} \frac{\partial}{\partial z} \right).
\]  
\[(62) \]

In the absence of eddies, \( V = [v] \), which, again, is expected to be small, and \( W = [w] \). Substitution shows that \( V \) and \( W \) satisfy a continuity equation analogous to (58). Use of (61) and (62) to eliminate \([v]\) and \([w]\) in favor of \( V \) and \( W \) allows us to rewrite (55) and (57) as:

\[
\frac{\partial [M]}{\partial t} + \frac{V}{a} \frac{\partial [M]}{\partial \phi} + W \frac{\partial [M]}{\partial z} - a \cos \phi \mathcal{F}_x = \frac{1}{\rho_s} (\nabla \cdot \mathbf{EPF}),
\]

\[(63)\]

and

\[
\frac{\partial [\theta]}{\partial t} + \frac{V}{a} \frac{\partial [\theta]}{\partial \phi} + W \frac{\partial [\theta]}{\partial z} - [Q] = -\frac{1}{\rho_s \partial z} \left\{ \frac{\rho_s [v^* \theta^*]}{a} \frac{\partial [\theta]}{\partial \phi} + \frac{\rho_s [w^* \theta^*]}{a} \frac{\partial [\theta]}{\partial z} \right\},
\]

\[(64)\]

respectively, where

\[
\mathbf{EPF} \equiv \left[ 0, (EPF)_\phi, (EPF)_z \right]
\]

\[(65)\]

is the “Eliassen-Palm flux,” whose components are

\[
(EPF)_\phi = \rho_s \left\{ \frac{\partial [M]}{\partial z} \frac{[v^* \theta^*]}{\partial \theta} - \frac{[M^* v^*]}{\partial \theta} \right\},
\]

\[(66)\]

and

\[
(EPF)_z = \rho_s \left\{ -\frac{1}{a} \frac{\partial [M]}{\partial \phi} \frac{[v^* \theta^*]}{\partial \theta} - \frac{[M^* w^*]}{\partial \theta} \right\}.
\]

\[(67)\]
In (66), the $- [M' v']$ term is usually dominant, and in (67) the $- [v' \theta']$ term is usually dominant. Compare (66) and (67) with (17) and (47). When the EPF points upward, the meridional energy flux is in control. When it points in the meridional direction, the meridional flux of zonal momentum is in control. From (63) we see that a positive Eliassen-Palm flux divergence tends to increase $[M]$.

You should recognize the terms in the numerator inside $\frac{\partial}{\partial z}$ on the right-hand side of (64), the equation governing $[\theta]$, as the gradient-production terms of the equation for the eddy potential temperature variance, which was discussed in Chapter 7.

The preceding derivation appears to be nothing more than an algebraic shuffle. We wrote down (61) and (62) without any explanation or motivation. What is the point of all this? The point is that for steady linear waves with $F_x = F_y = 0$ and $Q = 0$, it can be shown that

$$\nabla \cdot (\text{EPF}) = 0. \quad (68)$$

Recall that this follows essentially from $[v' q'] = 0$. It turns out that the eddy forcing term of (64) is zero under the same conditions, i.e.

$$\frac{\partial}{\partial z} \left\{ \rho_s [v' \theta'] \frac{1}{a} \frac{\partial \theta}{\partial \phi} + \rho_s [w' \theta'] \frac{\partial \theta}{\partial z} \right\} = 0. \quad (69)$$

This follows essentially from our assumptions that: 1) $[M]$ does not change, and 2) gradient wind balance is maintained.

For the case of steady, linear waves, in the absence of friction and heating, our system of equations reduces to

---

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\[
\frac{\partial [M]}{\partial t} + \frac{V}{a} \frac{\partial [M]}{\partial \phi} + W \frac{\partial [u]}{\partial z} = 0,
\]
\[
[u] \left( f + \frac{[u]}{a} \tan \phi \right) + \frac{1}{a} \frac{\partial \phi}{\partial \phi} = 0,
\]
\[
\frac{\partial [\theta]}{\partial t} + \frac{V}{a} \frac{\partial [\theta]}{\partial \phi} + W \frac{\partial [\theta]}{\partial z} = 0,
\]
\[
\frac{1}{a \cos \phi} \frac{\partial}{\partial \phi} (\rho S v \cos \phi) + \frac{\partial}{\partial z} (\rho S W) = 0,
\]
\[
\frac{\partial [\phi]}{\partial z} - \frac{R[\theta]}{H} e^{\frac{-k}{H}} = 0.
\]

(70)

This system has the following steady solution:

\[
\frac{\partial [M]}{\partial t} = 0, \quad M \text{ in gradient-wind balance},
\]
\[
V = 0, \quad W = 0,
\]
\[
\frac{\partial [\theta]}{\partial t} = 0, \quad [\theta] \text{ specified from the past history or radiative-convective equilibrium}.
\]

(71)

This is very similar to the idealized solution with no mean meridional circulation that we discussed in Chapter 5. (There \([\theta]\) was determined by the specified “equilibrium” distribution, \(\theta_E\).) From the definitions of \(V\) and \(W\), we can solve for the mean meridional circulation implied by \(V = 0\) and \(W = 0\):

\[
\rho_S [v] = \frac{\partial}{\partial z} \left( \rho_S \frac{\partial [\theta^*]}{\partial \phi} \right),
\]

(72)

\[
[w] = \frac{1}{a \cos \phi} \frac{\partial}{\partial \phi} \left( \cos \phi \frac{\partial [\theta^*]}{\partial \phi} \right).
\]

(73)

As an aid for the interpretation of these results, suppose we had a solution with no eddies at all. This is essentially the same solution that we discussed in Chapter 5, for the case of no friction. “No eddies” certainly qualify as “steady linear waves,” so the above argument applies,
and from (72) and (73) we conclude that the MMC vanishes in the absence of eddies. In other words, the MMC is eddy-driven.

Now add steady linear eddies, so that \( \nabla \cdot (\mathbf{EPF}) \) continues to be zero. Then exactly the same \( [M] \) and \( [\theta] \) will satisfy the equations! Of course, \( V \) and \( W \) will be different, i.e., the MMC will be different, because it will have to be just what it takes to ensure that \( V=W=0 \), i.e., to satisfy (72) and (73). We can say that this MMC is “induced” by the eddies. The system produces this MMC in order to prevent the eddies from disrupting the thermal wind balance. Perhaps a better way to say this is that the processes that act to maintain thermal wind balance (i.e., geostrophic and hydrostatic adjustment) accomplish this feat by using the “wave-induced” MMC as a tool.

The interpretation of this amazing result is that if you try to modify \( [M] \) and \( [\theta] \) by applying eddy forcing such that \( \nabla \cdot (\mathbf{EPF}) = 0 \) (no potential vorticity flux), you will be disappointed! All that will happen is that the MMC will change, in such a way that \( V \) and \( W \) continue to be zero. In effect, the eddies will induce an MMC that exactly cancels the direct effects of the eddies on \( [M] \) and \( [\theta] \).

When the eddies are unsteady, the residual circulation is different from zero, and \( [u] \) and \( [\theta] \) are modified by the combined effects of the eddies and/or the eddy-induced MMC. Cancellation of the effects of the eddies and the MMC still tends to occur, but the cancellation is incomplete.

Edmon et al. (1980) discussed the quasi-geostrophic form of the non-interaction theorem, and used it to analyze the data of Oort and Rasmussen (1971). As a reminder [see (47)], the meridional component of the quasi-geostrophic \( \mathbf{EPF} \) is

\[
(EPF)_\phi = -a \cos(\phi) \left[ u^* v^* \right],
\]

and the vertical component is

\[
(EPF)_p = fa \cos(\phi) \frac{\partial (v^* \theta^*)}{\partial p}.
\]

[Compare (74) and (75) with (66) and (67), respectively.] Fig. 9.10 shows the contributions of the transient eddies to the Eliassen-Palm fluxes. First consider the winter results, shown in the upper panel. Near the surface in middle latitudes, we see arrows pointing strongly upward, indicating an intense poleward potential temperature flux. Near the tropopause, the arrows curve over and become horizontal, pointing towards the tropics. This indicates a strong poleward eddy...
momentum flux. The contours in the figure show the divergence of the Eliassen-Palm flux. Keep in mind that $\nabla \cdot \text{EPF} > 0 \text{ means } \frac{\partial [M]}{\partial t} > 0$, i.e., a positive EPF divergence favors westerly acceleration. The negative divergence (i.e., convergence) near 200 mb at about 30° N indicates that the net effect of the eddies is to decelerate the jet. In fact, the westerlies are being decelerated throughout middle latitudes, except near the surface. Note that this EPF convergence results mainly from the upward decrease of the upward component of the flux, i.e., it is mainly due to the energy flux.
The results for summer are quite similar, except that the action is generally weaker, and shifted poleward.

Figure 9.10: Contribution of transient eddies to the seasonally averaged Eliassen-Palm cross sections for the troposphere: (a) 5-year average from Oort and Rasmusson (1971) for winter; (b) the same for summer. The contour interval is $20 \times 10^{15} \text{ m}^3$ for (a), and $1 \times 10^{15} \text{ m}^3$ for (b). The horizontal arrow scale for the horizontal component in units of m$^3$ is indicated at bottom right; note that it is different from diagram to diagram. A vertical arrow of the same length represents the vertical component, in m$^3$ kPa, equal to that for the horizontal arrow multiplied by 80.4 kpa. From Edmon et al. (1980).
Fig. 9.11 shows the corresponding results for the stationary waves. In winter, the “strong” arrows are pointing nearly straight up everywhere, indicating that the poleward eddy potential temperature flux is playing a much more important role than the eddy momentum flux. The westerlies are decelerated aloft, near 50° N, but they are accelerated near the surface. In summer the arrows point downward. The eddy momentum flux is important near the summer tropopause, but again the eddy potential temperature flux is more important overall. The westerlies are...
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strongly decelerated near the surface in the subtropics, and they are actually accelerated at 200 mb near 35° N.

Figure 9.12: Total (transient plus stationary) Eliassen-Palm cross sections for the troposphere: (a) 5-year average from Oort and Rasmusson (1971) for winter; (b) the same, respectively, for summer. The contour interval is $2 \times 10^{15}$ m$^3$ for (a), and $1 \times 10^{15}$ m$^3$ for (b). The horizontal arrow scale in units of m$^3$ is indicated at bottom right. From Edmon et al. (1980).
Fig. 9.12 shows the combined effects of the transient and stationary eddies. Note that the transient eddies dominate, in both seasons. Finally, Fig. 9.13 shows the residual circulation, $(V,W)$, for summer and winter. In winter, the residual circulation looks suspiciously like a giant Hadley Cell, extending from the tropics to the poles. This is reminiscent of the mean meridional circulation as seen in isentropic coordinates. In summer, we seem to see the northern edge of a
Hadley Cell extending into the Southern Hemisphere. Clearly, we can regard the residual circulation as a response to heating.

**Angular momentum and wave energy fluxes isentropic coordinates**

The Eliassen-Palm theorem is somewhat simpler and easier to interpret when we use isentropic coordinates. Following Andrews (1983), we begin with the flux form of the angular momentum equation in isentropic coordinates:

\[
\frac{\partial}{\partial t} (\rho_\theta u M) + \frac{1}{a \cos \phi} \frac{\partial}{\partial \lambda} (\rho_\theta u M) + \frac{\partial}{\partial \phi} (\rho_\theta v M \cos \phi) = -\rho_\theta \frac{\partial s}{\partial \lambda} - \frac{\partial}{\partial \theta} (\rho_\theta \theta M) - F_\lambda a \cos \phi.
\]

(76)

Recall that \( \rho_\theta \) is the pseudo-density:

\[
\rho_\theta = -\frac{1}{g} \frac{\partial p}{\partial \theta},
\]

(77)

and

\[
s \equiv c_p T + gz = \Pi \theta + gz,
\]

(78)

where

\[
\Pi \equiv c_p \left( \frac{p}{p_0} \right)^\kappa
\]

(79)

is the Exner function. We also define \( F_\lambda \) as the frictional sink of zonal momentum. Using the hydrostatic equation in isentropic coordinates, i.e.,

\[
\frac{\partial s}{\partial \theta} = \Pi,
\]

(80)

we obtain
\[ p_\theta \frac{\partial s}{\partial \lambda} = -\frac{1}{g} \left( \frac{\partial p}{\partial \theta} + \frac{p}{\partial \lambda} \right) \frac{\partial s}{\partial \lambda} = -\frac{1}{g} \left( \frac{\partial}{\partial \theta} \left( \frac{p}{\partial \lambda} \right) + \frac{p}{\partial \lambda} \frac{\partial}{\partial \theta} \right) \]
\[ = -\frac{1}{g} \frac{\partial}{\partial \theta} \left[ \frac{p}{\partial \lambda} \left( \Pi + gz \right) \right] + \frac{p}{\partial \lambda} \frac{\partial \Pi}{\partial \theta} \]
\[ = -\frac{1}{g} \frac{\partial}{\partial \theta} \left( \theta \frac{\partial \Pi}{\partial \lambda} + pg \frac{\partial z}{\partial \lambda} \right) + \frac{p}{\partial \lambda} \frac{\partial \Pi}{\partial \theta} \]
\[ = -\frac{\partial}{\partial \theta} \left( \frac{p}{\partial \lambda} \frac{\partial \Pi}{\partial \lambda} + \frac{p}{\partial \lambda} \frac{\partial z}{\partial \lambda} \right) . \]

(81)

This is a very peculiar result. It says that the effect of the horizontal pressure-gradient force on the angular momentum can be written as the divergence of a vertical flux of angular momentum given by \( \left( \frac{p}{\partial \lambda} \frac{\partial \Pi}{\partial \lambda} + \frac{p}{\partial \lambda} \frac{\partial z}{\partial \lambda} \right) \). The pressure-gradient force in the meridional equation of motion (multiplied by the pseudo-density) can be written in a similar way.

Substituting (81) back into (76) gives the angular momentum equation in the form
\[
\frac{\partial}{\partial t} \left( \rho_\theta M \right) + \frac{1}{a \cos \phi} \left\{ \frac{\partial}{\partial \phi} \left( \rho_\theta u M \cos \phi \right) + \frac{\partial}{\partial \phi} \left( \rho_\theta v M \cos \phi \right) \right\} = \frac{\partial}{\partial \theta} \left( \frac{p}{\partial \lambda} \frac{\partial \Pi}{\partial \lambda} + \frac{p}{\partial \lambda} \frac{\partial z}{\partial \lambda} - \rho_\theta \theta M \right) - F_\lambda a \cos \phi .
\]

(82)

When we take the zonal mean of (82), the \( \frac{\partial}{\partial \theta} \left( \frac{p}{\partial \lambda} \frac{\partial \Pi}{\partial \lambda} \right) \) term drops out (Why?), and we are left with
\[
\frac{\partial}{\partial t} \left[ \rho_\theta M \right] + \frac{1}{a \cos \phi} \left\{ \frac{\partial}{\partial \phi} \left[ \rho_\theta v M \right] \cos \phi \right\} = \frac{\partial}{\partial \theta} \left[ \rho_\theta M \frac{\partial z^*}{\partial \phi} \right] - \frac{\partial}{\partial \theta} \left[ \rho_\theta \theta M \right] - \left[ F_\lambda \right] a \cos \phi .
\]

(83)

The pressure-gradient term of (83) has a very simple and interesting form: It is proportional to the change with \( \theta \) of the zonal mean of the product of the pressure and the slope of the height of the isentropic surface. The expression \( \left[ p \frac{\partial z^*}{\partial \lambda} \right] \) can be interpreted as form drag on the isentropic surface, analogous to the form drag on mountains discussed earlier in the book. Here the “mountains” are upward bulges of the isentropic surfaces, associated with blobs of cold air at a given pressure level; the “valleys” are downward bulges of the isentropic surfaces, associated with blobs of warm air at a given pressure level. We can say that

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upward flux of zonal momentum due to the wave = \[-p^* \frac{\partial z^*}{\partial \lambda}\].

(84)

This shows that, from the perspective of isentropic coordinates, the upward flux of zonal momentum is associated with the pressure force, rather than with a covariance between the “vertical velocity” (which vanishes in isentropic coordinates in the absence of heating) and the zonal velocity. A layer of air confined between two isentropic surfaces will feel two momentum fluxes associated with the pressure force: one on its underside, and a second on its upper side. It is the difference between these two forces that tends to produce a net acceleration of the layer. That is why we see \(\frac{\partial}{\partial \theta} \left( p^* \frac{\partial z^*}{\partial \lambda} \right)\) in (83). This form of the vertical momentum flux was first discussed by Klemp and Lilly (1978).

The analogy between the form drag on a “wavy” isentropic surface and the form drag on topography is a very powerful aid to physical intuition. The form drag on a wavy isentropic surface is expected to be different from zero when the isentropic surface is moving relative to the mean flow, i.e., when \([u] - c \neq 0\), where \(c\) is the phase speed of the wave relative to the Earth’s surface. This is analogous to the fact that a form drag on topography is expected to be different from zero when there is a low-level mean flow relative to the Earth’s surface. A wavy isentropic surface moving to the east relative to the mean flow is expected to experience a form drag that pushes it back towards the west, i.e., that tries to slow it down relative to the mean flow. This corresponds to an upward flux of westerly momentum, because the air above the wavy surface is being pushed towards the east. Similarly, a wavy isentropic surface moving towards the west relative to the mean flow is expected to experience a form drag that pushes it back towards the west, and this corresponds to a downward flux of westerly momentum. \(The \ sign \ of \ the \ (positive \ upward) \ wave \ momentum \ flux \ is \ therefore \ expected \ to \ be \ opposite \ to \ the \ sign \ of \ [u] - c\). Based on this argument, Kelvin waves are expected to produce an upward flux of westerly momentum, and Rossby waves and Yanai waves are expected to produce a downward flux of westerly momentum.

Before completing our discussion of the Eliassen-Palm theorem in isentropic coordinates, it is useful to recall the form of the mechanical energy equation in isentropic coordinates, which was given in Chapter 4 and is repeated here for your convenience:

\[
\left[\frac{\partial}{\partial t}(\rho_0 K)\right]_0 + \nabla_\theta \cdot \left[ \rho_0 V(K + \phi) \right] + \frac{\partial}{\partial \theta} \left[ \rho_0 \hat{\theta}(K + \phi) - z \left( \frac{\partial p}{\partial t} \right)_\theta \right] + V \cdot F_v = -\rho_0 \omega \alpha - \rho_0 \delta.
\]

(85)

The \(\frac{\partial}{\partial \theta} \left[ -z \left( \frac{\partial p}{\partial t} \right)_\theta \right]\) term on the right-hand side of (85) represents the vertical transport of energy via “pressure-work.” The upward flux of wave energy is, therefore, given by

\[\text{An Introduction to the Global Circulation of the Atmosphere}\]
Recall that for a neutral wave with zonal phase velocity $c$, we can write

$$\frac{\partial}{\partial t} = -\frac{c}{\cos \phi} \frac{\partial}{\partial \lambda}.$$  

It follows that, for a neutral wave,

$$\text{upward wave energy flux} = -\frac{c}{\cos \phi} p^* \left( \frac{\partial z^*}{\partial \lambda} \right)_{\theta}.$$  

(86)

Comparing (83) with (86), we conclude that for a neutral wave

$$\text{upward wave energy flux} = \frac{c}{\cos \phi} \text{ times the upward wave angular momentum flux.}$$  

(87)

This shows that, for neutral waves propagating towards the east relative to the air, i.e., $c > 0$, the momentum flux and the energy flux have the same sign, while for neutral waves propagating towards the west relative to the air, i.e., $c < 0$, the momentum and energy fluxes have opposite signs. As we know, Rossby and Yanai waves always propagate west, so for Rossby waves and Yanai waves the momentum flux is always opposite in direction to the energy flux. Kelvin waves always propagate east, so for Kelvin waves the momentum flux is always in the same direction as the energy flux.

Stan and Randall (2007) explored the possibility of using potential vorticity as a meridional coordinate. They showed that the meridional flux of angular momentum can be expressed in terms of form drag along surfaces of constant potential vorticity.

**The Eliassen-Palm theorem in isentropic coordinates**

Now we relate the preceding analysis to the Eliassen-Palm theorem, following Andrews (1983) and Andrews et al. (1987). Recall that

$$[\rho_0^v A] = [\rho_0^v] [A] + \left[ \rho_0^v A \right],$$  

(88)

for an arbitrary variable $A$. Using (88) in the time-rate-of-change term of (83), we obtain

$$\frac{\partial}{\partial t} \left( [\rho_0^v] [M] \right) + \frac{1}{\cos \phi} \frac{\partial}{\partial \phi} \left( [\rho_0^v M] \cos \phi \right) =$$

$$-\frac{\partial}{\partial t} \left[ \rho_0^v M^* \right] + \frac{\partial}{\partial \theta} \left[ p^* \frac{\partial z^*}{\partial \lambda} \right] - \frac{\partial}{\partial \theta} \left[ \rho_0 M \hat{\theta} \right] - \left[ F_z \right] a \cos \phi.$$  

(89)
Here the “eddy part” of the time-rate-of-change term has been moved to the right-hand-side of the equals sign; this will be discussed later. We want to derive an “advective form” of (89), so we bring in the zonally averaged continuity equation in isentropic coordinates, which can be written as

\[
\frac{\partial \rho}{\partial t} + \frac{1}{a \cos \phi} \frac{\partial \left( \left( \rho v \right) \cos \phi \right)}{\partial \phi} = -\frac{\partial \left[ \rho \theta \right]}{\partial \theta}.
\]  

(90)

Eq. (90) was discussed already in Chapter 3. Subtract \( M \) times (90) from (89), to obtain

\[
\left[ \rho \theta \right] \frac{\partial [M]}{\partial t} + \frac{1}{a \cos \phi} \left\{ \partial \left( \left( \rho \theta v \right) \cos \phi \cos \phi \right) \right\} - \frac{\partial}{\partial \phi} \left( \left( \rho v \right) \cos \phi \right) = -\frac{\partial \left[ \rho \theta - \rho \theta \theta \right]}{\partial \theta} - \left[ F_\lambda \right] \cos \phi.
\]

(91)

We cannot yet combine the meridional and vertical derivative terms of (91) to obtain an advective form.

In order to obtain the desired advective form (a few steps further on), we need to introduce a mass-weighted zonal mean, defined by

\[
\hat{A} \equiv \frac{\rho \theta A}{\rho \theta}.
\]  

(92)

Using the definition (92), we can write

\[
\rho \theta A = \left[ \rho \theta A \right] + \rho \theta A^* = \left[ \rho \theta \right] \hat{A} + \rho \theta A^*.
\]  

(93)

and

\[
\left[ \rho \theta AB \right] = \left[ \rho \theta A \right] B + \left( \rho \theta A \right)^* B^* = \left[ \rho \theta \right] \hat{A} B + \left( \rho \theta A \right)^* B^*.
\]  

(94)

where \( B \) is a second arbitrary variable. As special cases of (94), we can write the zonally averaged meridional and vertical fluxes of \( B \) as
\[ [\rho_v B] = [\rho_v] [B] + [(\rho_v)^* B^*] \]
\[ = [\rho_v] [\hat{v}] [B] + [(\rho_v)^* B^*], \]

and

\[ [\rho_\theta \hat{B}] = [\rho_\theta \hat{\theta}] [B] + [(\rho_\theta)\theta^* B^*] \]
\[ = [\rho_\theta] [\hat{\theta}] [B] + [(\rho_\theta)\theta^* B^*]. \]

It is convenient to give names to some of these quantities. The “eddy meridional mass flux,” \((\rho_v)^*\) has a zonal mean of zero. This means that \((\rho_v)^*\) does not transport any mass on the average. It is a “mixing” or “diffusive” or “sloshing” mass flux. A similar comment applies to the eddy vertical mass flux, \((\rho_\theta)^*\).

Using (93) and (94), we can rewrite (90) and (91) as

\[
\frac{\partial}{\partial t} [\rho_v] + \frac{1}{\alpha \cos \phi} \frac{\partial}{\partial \phi} \left( [\rho_v] [\hat{v}] \cos \phi \right) = -\frac{\partial}{\partial \theta} \left( [\rho_v] [\hat{\theta}] \cos \phi \right)
\]

and

\[
\left[ \frac{\partial}{\partial t} \left[ \rho_a M^* \right] \right] + \frac{1}{\alpha \cos \phi} \frac{\partial}{\partial \phi} \left( \left[ \rho_v \right] [\hat{v}] \left[ M \right] + \left[ (\rho_v)^* M^* \right] \cos \phi \right) - \frac{\partial}{\partial \theta} \left( \left[ \rho_v \right] [\hat{\theta}] \left[ M \right] + \left[ (\rho_\theta)\theta^* M^* \right] \cos \phi \right) - [F_a] \alpha \cos \phi,
\]

respectively. The meridional and vertical derivatives can now be combined to obtain the desired advective form. We also divide by \([\rho_v]\), simplify, and rearrange, to obtain

\[
\frac{\partial}{\partial t} [M] + [\hat{v}] \frac{\partial}{\partial \phi} [M] + [\hat{\theta}] \frac{\partial}{\partial \theta} [M] = -\frac{1}{[\rho_v]} \frac{\partial}{\partial t} \left[ \rho_a M^* \right] + \frac{1}{[\rho_v]} \frac{\partial}{\partial \theta} \left[ p^* \frac{\partial z^*}{\partial \lambda} \right]
\]
\[ - \frac{1}{[\rho_v] \cos \phi} \frac{\partial}{\partial \phi} \left( [(\rho_v)^* M^*] \cos \phi \right) - \frac{1}{[\rho_v]} \frac{\partial}{\partial \theta} \left( (\rho_\theta)\theta^* M^* \right) - \frac{[F_a] \alpha \cos \phi}{[\rho_v]}.
\]

Here all of the eddy terms (and friction) have been collected on the right-hand side, and the remaining terms have been collected on the left-hand side.
Now define the isentropic Eliassen-Palm flux vector as

\[ \mathbf{EPF} \equiv (0, EPF_\varphi, EPF_\theta), \]

where

\[ EPF_\varphi \equiv -[(\rho_0 v)^* M^*], \] and \[ EPF_\theta \equiv \left[ p^* \frac{\partial z^*}{\partial \lambda} \right] - \left[ (\rho_0 \theta)^* M^* \right]. \]

(97)

The meridional component is minus the eddy angular momentum flux. The vertical component is minus the “total” vertical eddy angular momentum flux, due to the combination of isentropic form drag and the vertical mass flux associated with heating. The divergence of the isentropic Eliassen-Palm flux is given by

\[ \nabla \cdot \mathbf{EPF} = -\frac{1}{a \cos \varphi} \frac{\partial}{\partial \varphi} \left\{ \left[ (\rho_0 v)^* M^* \right] \cos \varphi \right\} + \frac{\partial}{\partial \theta} \left\{ \left[ p^* \frac{\partial z^*}{\partial \lambda} \right] - \left[ (\rho_0 \theta)^* M^* \right] \right\}, \]

(98)

where it is understood that the meridional derivative is taken along an isentropic surface. With these definitions, (96) can be written as

\[
\begin{array}{c}
\frac{\partial [M]}{\partial t} + \left[ \frac{v}{a} \right] \frac{\partial [M]}{\partial \varphi} + \left[ \frac{\partial}{\partial \theta} \right] \frac{\partial [M]}{\partial \theta} = \frac{1}{[\rho_0]} \left( -\frac{\partial [\rho_0 M^*]}{\partial t} + \nabla \cdot \mathbf{EPF} - \frac{F_\lambda}{[\rho_0]} a \cos \varphi \right). \\
\end{array}
\]

(99)

This derivation does not rely on an assumption of gradient wind balance, or any other form of balance. It is therefore more general than the results presented earlier.

Consider a steady state (or time average) with no heating. Then the continuity equation (90) reduces to

\[ \frac{\partial}{\partial \varphi} ([\rho_0 v] \cos \varphi) = 0, \text{ for steady flow without heating.} \]

(100)

Since \([\rho_0 v] \cos \varphi = 0\) at both poles, we conclude that

\[ [\rho_0 v] = 0 \text{ for all } \varphi, \text{ for steady flow without heating,} \]

(101)

from which it follows that
\[ \hat{v} = 0 \text{ for all } \varphi, \text{ for steady flow without heating.} \]  

(102)

In other words, when we see \[ \hat{v} \neq 0 \] in a time average, it is due to diabatic processes. This was discussed already near the end of Chapter 3.

Eq. (102) tells us that for steady flow with no heating the meridional advection term of (99) vanishes, which is quite amazing. Naturally the tendency term of (99) is also zero in this case. When friction is also negligible, it follows from (99) that the Eliassen-Palm flux is non-divergent:

\[ \nabla_\theta \cdot \mathbf{EPF} = 0 \text{ for steady flow without heating or friction.} \]  

(103)

Another way of saying this is that, for steady flow in the absence of heating and friction, the zonally averaged meridional transport of angular momentum is due only to the eddies, and is balanced by the form drag on isentropic surfaces. This beautifully simple result is pretty nearly exact. It is a statement of the Eliassen-Palm theorem.

With the isentropic system, there is no need to define a “residual” circulation, because \textit{the true zonally averaged circulation as seen in isentropic coordinates is the residual circulation}. This circulation vanishes for a steady state (or time average) with no heating, even when friction is present. The time-averaged mean meridional circulation in isentropic coordinates is due entirely to heating. This point was made near the end of Chapter 3.

\textbf{Potential vorticity fluxes}

In Chapter 4, we derived the potential vorticity equation in the form

\[ \frac{\partial (\rho \theta q)}{\partial t} + \nabla_\theta \cdot (\rho \theta \mathbf{V} q) = \nabla_\theta \cdot \left\{ \mathbf{k} \times \left( \frac{\partial \mathbf{V}}{\partial \theta} + \mathbf{F} \right) \right\}. \]  

(104)

Here we have put the diabatic and frictional terms on the right-hand side, even though they can be combined with the advection term on the left-hand side. Taking zonal average of (104), and collecting terms, we obtain

\[ \frac{\partial}{\partial t} \left[ \rho \theta q \right] + \frac{1}{a \cos \varphi} \frac{\partial}{\partial \varphi} \left( \left[ \rho \theta v q + \theta \frac{\partial u}{\partial \theta} + F_k \right] \cos \varphi \right) = 0. \]  

(105)

This form highlights the fact that there is a meridional flux of PV due to diabatic and frictional processes. Using (88), (93) and (94), we can rewrite (105) as

\[ \textit{An Introduction to the Global Circulation of the Atmosphere} \]
\[
\frac{\partial}{\partial t} [\rho \theta] + \frac{1}{a \cos \phi} \frac{\partial}{\partial \phi} \left\{ \left[ \rho \left[ \hat{v} \right] \right] [q] + \left[ (\rho \theta v)^* q^* \right] + \left[ \theta \frac{\partial u}{\partial \theta} \right] + \left[ F_\lambda \right] \right\} \cos \phi = 0.
\]

(106)

Applying a time average, (106) reduces to

\[
\frac{\partial}{\partial \phi} \left\{ \left[ \rho \theta \right] [q] + \left[ (\rho \theta v)^* q^* \right] + \left[ \theta \frac{\partial u}{\partial \theta} \right] + \left[ F_\lambda \right] \right\} \cos \phi = 0.
\]

(107)

This amazing result was derived by Haynes and McIntyre (1987; their Eq. 3.4). The quantity in curly brackets is independent of latitude. It is zero at both poles, because of the factor of \( \cos \phi \).

Therefore it must be zero at every latitude, which implies that

\[
\left[ \rho \theta \right] [q] + \left[ (\rho \theta v)^* q^* \right] + \left[ \theta \frac{\partial u}{\partial \theta} \right] + \left[ F_\lambda \right] = 0 \text{ for all } \phi.
\]

(108)

Using observations of the pseudo-density, the meridional wind, and the PV, Eq. (108) can be used to diagnose the zonally averaged diabatic and frictional fluxes of PV along isentropic surfaces.

With no heating or friction, and using (102), Eq. (108) reduces to

\[
\left[ (\rho \theta v)^* q^* \right] = 0 \text{ for all } \phi \text{ in a time average without heating or friction.}
\]

(109)

Compare with (16), which was derived for the quasi-geostrophic case. An interpretation of (109) is that a time-averaged meridional eddy flux of potential vorticity is only possible in the presence of heating and friction. Also compare with (103). The conditions for the eddy PV flux to vanish are the same as the conditions for the isentropic Eliassen-Palm flux to be non-divergent. Again, this is consistent with the quasi-geostrophic case discussed earlier.

**The quasi-biennial oscillation**

The zonal winds of the tropical stratosphere undergo an amazing “Quasi-Biennial Oscillation” (QBO), with a period of about 26 months. The QBO was discovered by Richard
Reed in about 1960 (see the early review by Reed, 1965). The early evidence was not sufficient to establish the existence of a true oscillation in a statistically significant fashion, but additional decades of data have made it clear that a quasi-periodic oscillation really exists, as shown in Fig. 9.14. The observations show that a “ring” of air in the tropical stratosphere, extending over all longitudes, reverses the direction of its zonal motion roughly every two years, like a giant zonally oriented Ferris Wheel. The winds shift from about 20 m s\(^{-1}\) westerly to 20 m s\(^{-1}\) easterly, and back again, so the changes are quite large. They are observed to propagate down from the middle stratosphere to near the tropopause. Corresponding oscillations are seen in other stratospheric fields, and much more weakly in the troposphere.

A theory of the QBO was proposed by Lindzen and Holton (1968) and Holton and Lindzen (1972). According to this theory, the oscillation is due to the interactions of upward-propagating Kelvin and Yanai waves with the mean flow. More recently, it has been proposed that eastward- and westward-propagating gravity waves play an important role, and the Kelvin and Yanai waves are now being de-emphasized.
The various equatorially trapped waves discovered by Matsuno (1966), which were discussed in Chapter 8, are observed to produce energy propagation upward into the stratosphere. The source of wave energy must, therefore, be in the troposphere, and is believed to be associated with latent heating.

Each type of wave can be blocked by a critical level where \( [u] - c = 0 \); here \( c \) is the phase speed of the wave. Eastward propagating waves, such as Kelvin waves and eastward-propagating inertia-gravity (EIG) waves (EIG), have critical levels in westerlies. Westward
propagating waves, such as Rossby waves, Yanai waves and westward-propagating inertia-
gavity (WIG) waves, have critical levels in easterlies.

As discussed earlier, because Kelvin waves propagate towards the east, relative to the
mean flow, the “isentropic form drag” paradigm tells us that upward propagating Kelvin waves
transport westerly momentum upward, i.e., they deplete the westerly momentum at lower levels,
and deposit this momentum aloft, thus tending to produce westerlies aloft and easterlies below.
Therefore upward propagating Kelvin waves tend to produce a westerly acceleration when they
encounter a critical level at the base of a layer of westerlies, thus causing the westerlies to
descend with time, as observed in the QBO. Recall that Kelvin waves do not involve fluctuations
of the meridional wind; because of this they produce no meridional eddy transports of any
variable. See Fig. 9.15.
Yanai waves, with \( c < 0 \), can propagate through westerlies, for which \( [u] - c > 0 \), but not through easterlies that are strong enough to make \( [u] - c < 0 \). Yanai waves that transport energy upward transport westerly momentum downward. The westward-propagating Yanai waves are damped in easterlies at the level where \( [u] - c = 0 \), and so they produce an easterly acceleration near the base of a layer of descending easterlies. Westward-propagating inertia-gravity waves will produce a similar effect, and recent work suggests that they are in fact important for the QBO.

Plumb (1984) produced a remarkable laboratory simulation of the QBO in an annular tank full of stratified salt water. In his experiment, “eastward” and “westward” propagating internal gravity waves are artificially excited by an oscillating diaphragm at the bottom of the tank. At a given level, the direction of the mean flow reverses periodically with time, and these reversals propagate downward. The oscillations are caused by wave-mean flow interactions. A video of this laboratory experiment is available on the class web page.

For many years, atmospheric general circulation models failed to simulate the QBO. Finally, however, M. Takahashi (1996) produced reasonably successful simulations of the QBO using a full atmospheric general circulation model of the troposphere and stratosphere. Earlier Carriolle et al. (1993) had produced a much weaker and less realistic but still promising simulation, and Takahashi and Shiobara (1995) had produced a QBO in a simplified GCM. During 1997, the ECMWF model began for the first time to produce a QBO when run in climate-simulation mode, i.e., without data assimilation. This improvement in the model’s performance was associated with an increase in the vertical resolution. It appears that high vertical resolution and possibly also weak damping are needed for a successful simulation of the QBO.

There is some evidence for phenomena similar to the QBO in the atmospheres of Jupiter (Orton et al., 1991; 1994; Leovy et al., 1991; Friedson, 1999; Flasar et al., 2004) and Saturn (Fouchet et al., 2008).

**Blocking**

Blocking (Berggren, 1949; Rex, 1950 a, b), is a low-frequency, mid-latitude phenomenon characterized by a nearly stationary anticyclone that persists for at least several days and sometimes up to several weeks. The anticyclone splits the westerly jet and steers cyclonic disturbances around itself, mainly on the poleward side. The flow in the vicinity of a blocking high is strongly meridional, and is an example of what the older literature calls a “low-index” regime (Rossby, 1939); a high-index regime is strongly zonal. Blocking highs tend to fluctuate in intensity, weakening briefly and then re-intensifying. They remain nearly stationary in an average sense, although they may wander around a little over their lifetimes.

Blocks tend to occur preferentially in the Northern Hemisphere, especially in the eastern North Atlantic and the eastern North Pacific in winter, and northern Asia in summer. Southern Hemisphere blocking does occur, most commonly near New Zealand. Blocking can occur in both summer and winter, but is more common in winter. Wintertime blocking events sometimes
appear to be associated with stratospheric sudden warmings (Quiroz, 1986). The formation of a block can be associated with upward propagation of a Rossby wave, which then interacts with the stratospheric zonal flow to produce a breakdown of the polar night vortex, i.e., a “Sudden Warming” event (e.g., Martius et al., 2009; Woolings et al., 2010).

Blocks strongly influence weather patterns for one or more weeks at a time, sometimes in association with heat waves and/or persistent droughts (e.g., Green, 1977). If the formation and dissipation of blocks can be predicted, then it will be possible to predict weather patterns with improved skill. Until recently, weather-prediction models were not very successful in forecasting blocks, and the same models produced blocks less often than observed when run in climate simulation mode. Within the last few years, this situation has improved dramatically, apparently as a result of increased model resolution.

What causes the formation of a blocking anticyclone? As discussed by Colucci (1985), blocks sometimes begin when particularly intense cyclones advect air with low potential vorticity (PV) from the subtropics (where low PV is the norm) into middle latitudes (where low potential vorticity is an anomaly). An example of this is shown in Fig. 9.16, which is taken from
Figure 9.16: Contours of the fourth root of the Ertel potential vorticity on the 320 K isentropic surface, which slopes from 200 mb near the pole to 600 mb in the tropics. Panels a - e are for successive days. The contour interval is 0.001 in SI units. The bold contour is 0.005. From Shutts (1986).
Shutts (1986). At the same time the air with low PV is advected poleward, a mass of high PV air is advected into position on the equatorward side of the low-PV anomaly. The block thus has a dipole structure in terms of PV. Near the longitude of the block, the PV decreases poleward, whereas it normally increases poleward. The blocking anticyclone could be called a “cut-off high,” while the cyclone is a “cut-off low.” Fig. 9.17 shows the corresponding sea level pressure and 500 mb height fields. The dipole structure is clearly evident in the latter. Note that the westerlies have “split” into a strong branch to the north of the high, and a weaker branch to the south of the low.

How is it possible for blocking highs to persist as well defined, isolated “objects,” even while they are embedded in the turmoil of the midlatitude winter circulation? The destruction of the low seems natural; the persistence of the high demands an explanation. Hoskins et al. (1985) speculated that lows are disrupted by convection, while highs can survive because convection is suppressed there.
Observations suggest that the smaller-scale transient eddies that are steered around a blocking high actually help to maintain the high (e.g., Hansen and Chen, 1982; Hoskins et al., 1983; Egger et al., 1986; Shutts, 1986; Dole, 1986; Mullen, 1987; Nakamura et al., 1997; Chang et al., 2002; Luo and Chen, 2006; Ren et al., 2009). Blocks may be examples of up-scale energy transport, which, as discussed in Chapter 10, is expected on theoretical grounds in two-dimensional turbulence. It has been suggested that blocking anticyclones in the Earth’s atmosphere are dynamical cousins of the “Great Red Spot” that has persisted in the atmosphere of Jupiter for at least several hundred Earth years.

The nearly stationary character of blocks suggests that they are somehow “anchored” to fixed geographic features such as topography, although Hu et al. (2008) reported a simulation of blocks with an “aqua-planet” model that uses zonally symmetric boundary conditions. A theory that involves topographic anchoring was proposed by Charney and DeVore (1979; hereafter CDV), who suggested that in the presence of topography the large-scale circulation can adopt either of two equilibrium states, one of which corresponds to blocking, and the other to a more zonal flow. The basic idea is that when the configuration of the mean flow is “right” there can be stationary waves that are resonantly forced by the topography. The waves feed back on the mean flow, however. Under certain conditions, the resonant waves alter the mean flow in a way that favors the formation of resonant waves, i.e., the waves create a mean flow that is favorable for their own continuing existence. On the other hand, if the waves are not present, the resulting mean flow is not favorable for the formation of the waves. Hence the system can exist in either of two possible configurations. These are referred to as “multiple equilibria.” Because the CDV theory involves the interactions of the waves with the mean flow, it is inherently nonlinear.

A simplified version of the CDV theory is as follows. The topography is assumed to vary sinusoidally in longitude, i.e.,

\[ h(x) \equiv h_T \cos(K_m x). \]  

(110)

The motion is described by a stream function, \( \psi \), which is assumed to be of the form

\[ \psi(x, y, t) = -U(t)y + A(t)\cos(K_m x) + B(t)\sin(K_m x). \]  

(111)
Think of this as a highly truncated functional expansion. See Fig. 9.18. Recall that the stream function is defined by the relations 
\[ u \equiv -\frac{\partial \psi}{\partial y} \quad \text{and} \quad v \equiv \frac{\partial \psi}{\partial x} \]; using these formulae, we see that the zonal component of the flow described by (111) is simply \( U(t) \), i.e., it depends only on time. The “A” part of the wave is in phase with the topography, in the sense that for \( A > 0 \) the maximum of the stream function (corresponding to ridging behavior) occurs over the mountain. On the other hand, for \( B > 0 \) the “B” part of the wave represents a trough downstream of the mountain. The wave-like meridional component of the flow varies with both \( x \) and \( t \):

\[
v(x,t) = K_m \left\{ -A(t) \sin(K_m x) + B(t) \cos(K_m x) \right\}.
\]

(112)

By substituting (111) into a suitable nonlinear vorticity equation describing a Rossby wave flow with friction, CDV showed that the wave motion satisfies

\[
\frac{1}{K_m} \frac{dA}{dt} + \left( \frac{v}{K_m} \right) A + \left( U - \frac{\beta}{K_m^2} \right) B = 0,
\]

(113)

Figure 9.18: The relationships between the \( A \) and \( B \) components of the meridional wind and the topography in the model of Charney and Devore. The topography is indicated by the wavy line in the center. The arrows indicate the direction of the \( v \)-wind component, with “up” corresponding to “southerly.” The arrows above the topography are for the \( B \)-component of the wave, and the arrows below are for the \( A \)-component, assuming that \( A \) and \( B \) are positive. The indicated regions are cyclonic or anticyclonic for the case of \( f > 0 \) (the Northern Hemisphere).
while the zonal flow obeys

\[
\frac{1}{K_m} \frac{dB}{dt} - \left( U - \frac{\beta}{K_m^2} \right) A + \left( \frac{\nu}{K_m} \right) B + \left( \frac{f_0 h_T}{K_m^2 H} \right) U = 0 ,
\]

(114)

In (113) - (114), the terms involving \( \nu \) represent friction. In (115), the “\( B \)” term represents the orographic form drag (or “mountain torque”) that the mountain exerts on the mean flow when the wave is oriented with the trough over the mountain, and the \( U^* \) term represents a “momentum forcing” that maintains the mean flow against friction. Note that (113) and (114) “blow up” if the wave number of the topography, \( K_m \), is equal to zero. This simply means that the wave solution does not occur in the absence of topography, i.e., the wave is topographically forced. Also note that (113) and (114) are nonlinear because they involve the products of \( A \) and \( B \) with \( U \). This nonlinearity represents the wave-mean flow interactions.

CDV considered equilibrium (steady) solutions of (113) - (115). These equilibria can be found by setting the time-rate of change terms to zero, solving the resulting linear system (113) - (114) for \( A \) and \( B \) as functions of \( U \), and selecting the appropriate value of \( U \) by requiring that the steady-state version of (115) also be satisfied. Fig. 9.19 shows an example, in which the straight line represents the \((B, U)\) pairs that satisfy (115), while the peaked line represents the \((B, U)\) pairs that satisfy (113) - (114). There are three equilibria, but it can be shown that the middle one is unstable. The stable equilibrium with large \( U \) has a small wave amplitude, and the stable equilibrium with small \( U \) has a large wave amplitude. This is understandable because the wave term of (115) represents a drag on \( U \). The solution with large wave amplitude and a weak zonal flow is interpreted as representing blocking. CDV suggested that if the model is subjected to stochastic forcing, representing the random fluctuations of the weather, perhaps manifested through fluctuations of \( U^* \), then the system can undergo occasional transitions back and forth between the neighborhood of the “blocked” equilibrium and the neighborhood of the “unblocked” equilibrium.
Charney and Straus (1980) extended the CDV theory to the baroclinic case, and it has been studied by many other authors. The CDV theory has been heavily criticized (e.g., Tung and Rosenthal, 1985), in part because its extreme idealizations limit the possible behaviors of the model, suggesting that the small number of discrete equilibria (i.e., two) is an artifact. Nevertheless the theory continues to be cited frequently as an important background concept for the interpretation of blocking, and also in research on the possible existence of multiple discrete weather “regimes” (e.g., O’Kane et al., 2013).

McWilliams (1980) suggested that “modons” could be considered as idealized models of blocks. Modons are exact solutions of the nonlinear vorticity equation (Flierl, 1978). They have dipole structures, in which a high (a negative vorticity center) is paired with a neighboring low (a positive vorticity center). Modons must have finite amplitudes in order to exist; there is no such thing as a “linear” modon. An interesting and block-like property of modons is that they are resistant to disruption by perturbations. In addition, a modon “translates” relative to the mean flow, and it is possible to set up a modon that is stationary relative to the Earth, in the presence of background westerlies. The conditions for this are special, however, and it is not clear why such special conditions should occur in real cases.
We can identify the following issues in connection with blocking, among others:

- What causes the formation of a blocking anticyclone?
- What determines the preferred geographical locations of blocking activity?
- How are blocking highs maintained against the noisy background flow?
- Why are blocks nearly stationary even though they are embedded in strong westerly currents?
- What causes the breakdown of a block?
- Why do we observe persistent, quasi-stationary anticyclones but not persistent, quasi-stationary cyclones?

The discussion given above shows that we have at least partial answers to some of these questions.

**Summary**

Waves and other eddies produce important effects on the large-scale circulation of the atmosphere. Important fluxes are associated with a wide variety of waves, including gravity waves, Rossby waves, and Kelvin waves. Momentum fluxes and temperature fluxes can tend to produce mutually counteracting effects, so that the mean zonal flow and temperature may not be altered. The appreciable effects of the eddies on the mean flow are typically associated with developing or decaying eddies, rather than steady, equilibrated eddies.

Two particularly important examples of wave mean-flow interactions in the general circulation are stratospheric sudden warmings and the quasi-biennial oscillation. Both involve wave propagation from the troposphere into the stratosphere.
Problems

1. Derive
\[
\begin{bmatrix} v^* q^- \end{bmatrix} = \nabla \frac{\partial}{\partial r} \left( H \frac{\partial}{\partial \rho} \left[ v^* T^* \right] \right)
\]
\[
= -\frac{\partial}{\partial y} \left[ u^* v^* \right] - \frac{\partial}{\partial \rho} \left( H \frac{\partial}{\partial \rho} \left[ v^* T^* \right] \right).
\]

(116)

2. Prove that for horizontally nondivergent flow on the sphere the vorticity flux and the angular momentum flux are related by
\[
\left[ v^* \zeta^* \right] \cos \phi = -\frac{1}{a \cos \phi} \frac{\partial}{\partial \phi} \left( \left[ v^* M^* \right] \cos \phi \right)
\]

(117)

Compare with (17).

3. Solve the one-dimensional steady-state version of the Charney-Devore model, as given by
\[
\left( \frac{v}{K_m} \right) A + \left( U - \frac{\beta}{K_m^2} \right) B = 0,
\]

(118)

\[
\left( U - \frac{\beta}{K_m^2} \right) A + \left( \frac{v}{K_m} \right) B + \left( \frac{f_0 h_T}{K_m H} \right) U = 0,
\]

(119)

\[
\left( \frac{f_0 h_T K_m}{4 H} \right) B - v(U - U^*) = 0.
\]

(120)

Make a plot like that shown in Fig. 9.19. In order to do this you will have to choose appropriate values for the various parameters of the model. Explain your choices.

4. Show that
\[
\psi^* z = \frac{g}{f_0} T^*.
\]

(121)

5. Using the data on isentropic surfaces for January 1, 2013 0GMT, available on the class web site, calculate and plot the following quantities for \( \theta \leq 600 \) K:
a) \[ \rho_o \]

b) \[ \rho_o \hat{\mathbf{v}} \] and \[ \rho_o \mathbf{v}^* \]

c) \[ q \]

d) \[ (\rho_o \mathbf{v})^* q^* \]

e) \[ (\rho_o \mathbf{v})^* M^* \]

f) \[ p^* \frac{\partial z^*}{\partial \lambda} \]

Caution: For reasons unknown (to me), the potential vorticity only is given on one “extra” theta surface, namely 320 K. All of the other fields are given on the same set of surfaces.

**QuickStudies Referenced**

Vertical Coordinate Systems