6. An overview of the effects of radiation and convection

Basics of moist convection

In atmospheric science, the term “convection” refers to a buoyancy-driven circulation. In other fields this is sometimes called “natural convection.”

Before discussing moist convection, it is useful to briefly investigate dry convection. Consider the equation of vertical motion and the statement of approximate conservation of dry static energy, linearized with respect to a resting, horizontally uniform basic state:

\[
\bar{\rho} \frac{\partial w'}{\partial t} = -\frac{\partial p'}{\partial z} - \rho' g ,
\]

\[
\frac{\partial s'}{\partial t} = -w' \frac{\partial \bar{s}}{\partial z} .
\]

(1)

(2)

Here the overbars denote horizontal averages, and primes denote departures from those averages. The gravity term of (1) represents the effects of buoyancy. Eq. (2) describes dry adiabatic motion.

We assume for simplicity that \( \frac{\partial \bar{s}}{\partial z} \) is independent of height. Also for simplicity, we neglect the perturbation pressure term in (1), which usually acts to partially cancel the effects of buoyancy, and we use the approximation

\[
-\left( \frac{\rho'}{\bar{\rho}} \right) \equiv \frac{T'}{T} = \frac{s'}{c_p T} .
\]

(3)

Then (1) reduces to
\[
\frac{\partial w'}{\partial t} = \frac{g s'}{c_p T}.
\]

(4)

Eqs. (2) and (4) form a closed system. We look for solutions of the form

\[
w'(t) = w'(0) \text{Re}\{e^{\sigma t}\},
\]

\[
s'(t) = s'(0) \text{Re}\{e^{\sigma t}\},
\]

(5)

where \(\sigma\) may be either real or imaginary. Substituting, we find that for nontrivial solutions

\[
\sigma^2 = -\frac{g}{c_p T} \frac{\partial s}{\partial z}.
\]

(6)

For \(\frac{\partial s}{\partial z} < 0\), \(\sigma\) is real, and there is an exponentially growing solution with \(\sigma > 0\); this is dry convective instability. We say that

\[
\frac{\partial s}{\partial z} < 0
\]

(7)

is the criterion for dry convective instability. It can be seen from either (2) or (4) that in the exponentially growing solution, with \(\sigma > 0\), \(w'(t)\) and \(s'(t)\) have the same sign for all time, so that \(w's' > 0\), i.e., convection transports dry static energy upward. We will show in Chapter 7 that an upward temperature flux tends to lower the atmosphere’s center of gravity, i.e., it reduces the total potential energy of the atmospheric column. The reduction in potential energy coincides with a generation of convective kinetic energy through the work done by the buoyancy force, so that the total energy is conserved. The generation of convective kinetic energy through an upward flux of dry static energy can be seen directly by multiplying both sides of (4) by \(w'\).

For \(\frac{\partial s}{\partial z} > 0\), \(\sigma\) is imaginary, and the solutions are oscillatory; these are gravity waves. Their frequency, \(N\), satisfies \(N^2 = \frac{g}{c_p T} \frac{\partial s}{\partial z}\); this is called the Brunt-Väisälä frequency. Using the analysis given above as a starting point, you should be able to show that \(\text{Re}\{e^{\sigma t}\} = 0\) for a gravity wave, where the overbar represents an average over the period of the wave. This means that gravity waves do not transport dry static energy.
The analysis above shows that, in the absence of phase changes, convection and gravity waves are mutually exclusive; they cannot occur in the same place at the same time. We will see later that this conclusion does not necessarily apply when phase changes are allowed.

Up to this point, we have considered dry adiabatic motion. To analyze moist convection, we will assume saturated moist adiabatic motion. As discussed in Chapter 4, the moist static energy, $h$, is approximately conserved under moist adiabatic processes. We replace (2) by

$$\frac{\partial h'}{\partial t} = -w' \frac{\partial h}{\partial z}. \tag{8}$$

For saturated motion, the moist static energy must be equal to the saturation moist static energy, $h^*$, so we rewrite (6) as

$$\frac{\partial h^*'}{\partial t} = -w' \frac{\partial h^*}{\partial z}. \tag{9}$$

Next, we have to relate the buoyancy term of (4) to $h^*$. Recall that

$$h^* \equiv c_p T + gz + Lq^* (T, p), \tag{10}$$

where the saturation mixing ratio depends, as indicated, on temperature and pressure. Perturbations at fixed height, and at approximately fixed pressure, satisfy

$$h^* = s' (1 + \gamma), \tag{11}$$

where, in the linearization,

$$\gamma = \frac{L}{c_p} \left( \frac{\partial q^*}{\partial T} \right)_p \tag{12}$$

is evaluated using the mean-state temperature and pressure. The nondimensional parameter $\gamma$ is positive and of order one.

We now write

$$-\left( \frac{\rho'}{\rho} \right) = \frac{T'}{T} = \frac{h^*'}{c_p \overline{T} (1 + \gamma)}. \tag{13}$$
This is analogous to (3). Substitution of (13) into the equation of vertical motion gives

\[
\frac{\partial w'}{\partial t} = \frac{gh''}{c_p\bar{T}(1+\gamma)}.
\]

(14)

We look for exponential solutions of the system (9) and (14), and find that

\[
\sigma^z = -\frac{g}{c_p\bar{T}(1+\gamma)} \frac{\partial h^*}{\partial z}.
\]

(15)

This shows that the criterion for moist convective instability of a saturated atmosphere is

\[
\frac{\partial h^*}{\partial z} < 0.
\]

(16)

Compare with (6).

Before moving on, we need to do one more thing. The dry adiabatic lapse rate of temperature is given by

\[
\Gamma_d = -\left(\frac{\partial T}{\partial z}\right)_{\text{dry adiabatic}} = \frac{g}{c_p}.
\]

(17)

This is the rate at which temperature decreases with height when the dry static energy is independent of height. We can rewrite (2) as

\[
\frac{\partial T'}{\partial t} = w'(\bar{T} - \Gamma_d)
\]

(18)

and re-state the criterion for dry convective instability as

\[
\Gamma > \Gamma_d.
\]

(19)

Similarly, we can express the criterion for moist convective instability in terms of the moist adiabatic lapse rate, which is given by
Eq. (20) is derived the QuickStudy on the moist adiabatic lapse rate. The denominator of (20) is larger than the numerator, so $\Gamma_m < \Gamma_d$, although $\Gamma_m \to \Gamma_d$ at cold temperatures. For example, with a pressure of 1000 mb and a temperature of 288 K, we find that $\Gamma_m = 4.67 \text{K km}^{-1}$ (see Fig. 6.1).

6.1). As the temperature increases, the moist adiabatic lapse rate decreases. For saturated motion, we can rewrite (2) as
\[
\frac{\partial T'}{\partial t} = w' \left( \bar{T} - \Gamma_m \right).
\]

(21)

Compare with (18). Eqs. (18) and (21) will be used later.

**Convective energy transports**

Riehl and Malkus (1958; Fig. 6.2) argued from the observed energy balance and vertical structure of the tropical atmosphere that deep, penetrative cumulus convection is the primary mechanism for upward energy transport in the tropics. They began by estimating the mass circulation across a latitude 10° on the winter side of the intertropical convergence zone (ITCZ). Recall that the “body” of the main solstitial Hadley Cell lies in the winter hemisphere. They neglected the mass transport across the boundary of the ITCZ on the summer side. They then attempted to evaluate the lateral energy transports across into and out of the ITCZ, as functions of height. They considered transports of internal energy, potential energy, and latent energy. Because the low-level inflow is warm and wet, while the upper level outflow is cold and dry, both internal and latent energy flow into the ITCZ; nevertheless there is a net loss of total energy due to the export of potential energy in the elevated outflow layer. The net export of energy by the meridional flow implies that there is a compensating net input of energy at the top and bottom of the column. Their estimates of the various quantities are summarized in Table 6.1.
Because energy flows into the ITCZ at low levels, and out at high levels, Riehl and Malkus concluded that there must be a net upward transport of energy inside the ITCZ. They argued, however, that this upward energy flux cannot be due to the mean flow, because the observed profile of moist static energy has a minimum at mid levels, as discussed in Chapter 3. If the mean flow was acting alone, then since is conserved following parcels would be uniform with height throughout the ascending column. Similar reasoning shows that diffusive energy transport cannot explain the observed upward energy flux. Riehl and Malkus concluded that the upward energy transport must occur in deep convective clouds that penetrate through the troposphere.

Table 6.1: Lateral energy transports on the poleward side of the ITCZ. Adapted from Riehl and Malkus (1958).

<table>
<thead>
<tr>
<th>δp</th>
<th>ν</th>
<th>$M_0$</th>
<th>s</th>
<th>$sM_0$</th>
<th>Lq</th>
<th>Lq$M_0$</th>
<th>hM</th>
<th>Eddy moisture transport, $10^{16}$ J s$^{-1}$</th>
<th>Total energy transport out of the ITCZ, $10^{16}$ J s$^{-1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10-9</td>
<td>-1.3</td>
<td>-5.2</td>
<td>301.5</td>
<td>-1.56</td>
<td>37.6</td>
<td>-0.20</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>9-8</td>
<td>-1.1</td>
<td>-4.4</td>
<td>305.7</td>
<td>-1.34</td>
<td>27.6</td>
<td>-0.12</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8-7</td>
<td>-0.4</td>
<td>-1.6</td>
<td>311.6</td>
<td>-0.49</td>
<td>18.4</td>
<td>-0.03</td>
<td></td>
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</tr>
<tr>
<td>7-6</td>
<td>0</td>
<td>0</td>
<td>317.8</td>
<td>0</td>
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<tr>
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<td>0</td>
<td>7.1</td>
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<td></td>
<td></td>
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<tr>
<td>5-4</td>
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<td>1.2</td>
<td>329.1</td>
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<td>335.4</td>
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<td>2.1</td>
<td>0.00</td>
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</tr>
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<td>5.2</td>
<td>340.8</td>
<td>1.77</td>
<td>0.8</td>
<td>0.00</td>
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<td>2.4</td>
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<td>0.06</td>
<td>0.07</td>
<td>0.13</td>
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An Introduction to the General Circulation of the Atmosphere
Neelin and Held (1987) considered the moist static energy budget of the ITCZ from a perspective very similar to that of Riehl and Malkus (1958). They reasoned as follows: In a time average, the vertically integrated moist static energy budget is expressed by

\[
g^{-1} \nabla \cdot \left( \int_0^{p_s} V h dp \right) = -\left( N_S - N_T \right),
\]

(22)

where \( N \) is the net downward flux of energy due to turbulence, convection and radiation, and subscripts \( T \) and \( S \) denote the top of the atmosphere and the surface, respectively. Similarly, mass continuity gives

\[
\nabla \cdot \left( \int_0^{p_s} V dp \right) = 0.
\]

(23)

Divide the column into upper and lower portions, and write

\[
p_s^{-1} \nabla \cdot \left( \int_0^{p_s} V h dp \right) = \nabla \cdot (V h)_u + \nabla \cdot (V h)_l = -g \left( \frac{N_S - N_T}{p_s} \right),
\]

(24)

\[
p_s^{-1} \nabla \cdot \left( \int_0^{p_s} V dp \right) = \nabla \cdot V_u + \nabla \cdot V_l = 0.
\]

(25)

In the tropics, horizontal variations of \( h \) are weak, so that it is useful to define \( h_u \) and \( h_l \) by

\[
\nabla \cdot (V h)_u \equiv h_u (\nabla \cdot V_u),
\]

and

\[
\nabla \cdot (V h)_l \equiv h_l (\nabla \cdot V_l),
\]

(26)

from which it follows that

\[
\nabla \cdot (V h)_u + \nabla \cdot (V h)_l = -\nabla \cdot (V_l)(h_u - h_l).
\]

(27)

Neelin and Held called the quantity \( h_u - h_l \) the “gross moist stability.” Eq. (27) shows that low-level convergence (i.e., \( \nabla \cdot V_l < 0 \), as in the ITCZ) leads to a net vertically integrated divergence.
of moist static energy (i.e., $\nabla \cdot (\nabla h)_u + \nabla \cdot (\nabla h)_l > 0$) when the gross moist stability is positive. This must be the average state of affairs in the tropics, because the tropical atmosphere exports moist static energy towards higher latitudes. In a region where the gross moist stability is negative, low-level convergence leads to a net vertically integrated convergence of moist static energy. If the troposphere is moistened, without changing the temperature profile, the gross moist stability tends to decrease.

We can also show from (24)-(27) that

$$\nabla \cdot \mathbf{V}_l = \frac{g}{p_s} \left( \frac{N_s - N_T}{h_u - h_l} \right)$$

(28)

This implies that low-level convergence ($\nabla \cdot \mathbf{V}_l < 0$) must occur where the column is gaining energy ($N_s - N_T < 0$), provided that

$$h_u - h_l > 0.$$  

(29)

According to (28), the pattern of the gross moist stability is closely linked to the pattern of low-level convergence, for a given distribution of $N_T - N_s$. As discussed later [see (48)], we expect the gross moist stability to be small where cumulus convection is active.

The term “gross moist stability” has been redefined in a variety of ways in work that followed that of Neelin and Held (1987). If you run across the term in a paper, check carefully to see what definition is being used.

**Radiative-convective equilibrium**

The study of Riehl and Malkus showed that in the tropics (and also in the moist convective regions of the summer hemisphere middle latitudes) upward transport of energy is due to small-scale convection rather than vertical advection by the large-scale vertical motion. Recall that the brightness temperature of the Earth corresponds to a level in the middle troposphere; the brightness temperature of the tropical convective regions corresponds to the actual temperature in the upper troposphere. We can consider that convection transports energy upward, to the middle or upper troposphere, where radiation can take over and carry the energy on out to space.

The simplest model that can represent this process is called a radiative-convective model. The basic idea is very simple. First, assemble physical parameterizations that suffice to determine the time rates of change of the temperature within the atmospheric column and at the Earth’s surface. Combine them into a model, and integrate using a time step on the order of an hour or so. Repeat until a steady state is approached to sufficient accuracy. Depending on the
initial conditions, convergence can take on the order of 500 simulated days. No significance is
ascribed to the time evolution itself; only the steady state is of interest. It is not obvious a priori
that a radiative-convective model will actually approach a steady state, but the models discussed
below do.

The physical ingredients of a radiative-convective model include parameterizations of
radiation, convection, turbulence, and the processes that determine the change of the surface
temperature. We write

\[ \rho c_p \frac{\partial T}{\partial t} = \rho L C - \frac{\partial F}{\partial z} + Q_R. \]  

(30)

Horizontal and vertical advection are deliberately omitted in (30), because the purpose of a
radiative-convective model is to help us to understand what the atmosphere would look like in
their absence. In order to use (30), we have to determine the condensation rate and the
convective fluxes. This will entail consideration of the moisture budget. In addition, the vertical
distributions of water vapor and clouds are needed to determine \( Q_R \), which satisfies

\[ Q_R = \frac{\partial}{\partial z} (S - R). \]  

(31)

Here \( S \) is the net solar radiation (positive down), and \( R \) is the net terrestrial radiation (positive
up).

We also impose an energy budget for the Earth’s surface:

\[ C_s \frac{\partial T_S}{\partial t} = N_S(T_S). \]  

(32)

Here \( C_s \) is the effective “heat capacity” of the surface, \( T_S \) is the surface temperature, and
\( N_S(T_S) \) denotes the net downward (i.e., a positive value denotes downward) vertical flux of
energy due to turbulence, convection, and radiation. Eq. (32) can be used to determine \( T_S \),
provided that the functional form of \( N_S(T_S) \) is specified. The value of \( C_s \) determines how
rapidly the surface temperature changes in response to a given value of \( N_S \). When \( C_s \) is large
\( T_S \) changes slowly. When \( C_s \rightarrow 0 \), \( T_S \) adjusts instantaneously, so as to keep \( N_S(T_S) = 0 \).

The radiative-convective balance requirements that must be satisfied in equilibrium can
be stated as follows:
As discussed in Chapter 2, there can be no net radiative energy flux at the top of the atmosphere:

\[ N_T = S_T - R_T = 0. \]  

(33)

Here \( N_T \) is also positive downward. The atmospheric column must be in energy balance, so

\[ N_S = N_T, \]  

(34)

where

\[ N_S = S_S - R_S - (F_h)_S \]
\[ = S_S + (R_S)_\downarrow - \sigma T_S^4 - (F_h)_S. \]  

(35)

From (33) - (34), it follows that, in equilibrium, the Earth’s surface must also be in energy balance:

\[ N_S = 0. \]  

(36)

We now discuss some results from radiative-convective models. In a series of studies during the 1960s (see bibliography), Manabe and his colleagues investigated the degree to which pure radiative equilibrium and/or radiative-convective equilibrium can explain the observed vertical distribution of temperature. These studies made major advances in our understanding of the vertical structure of the atmosphere. Because little was known about moist physics during the 1960s, Manabe et al. did not explicitly represent moist processes in their model; instead, they made alternative assumptions for the vertical distribution of moisture (discussed below), and adopted fairly drastic but empirically justified simplifying assumptions to determine the effects of latent heat release and moist convection on the atmospheric temperature profile.

They used a time-marching method and some simplifying assumptions to find equilibrium solutions. Here’s how that works: Suppose that “initial conditions” are specified for:

- the temperatures of the atmosphere and
- the temperature of the surface, and
- the water vapor content of the atmosphere, and
- the distribution of cloudiness as a function of height, and
- the composition of the dry air (including ozone) as a function of height.

They also specified the albedo of the Earth’s surface.
The vertical distribution of water vapor has a powerful effect on the radiation budgets of both the atmosphere and the surface. In order to predict the vertical distribution of water vapor, it is necessary to include in the model representations of turbulence and cumulus convection. Manabe and Wetherald did not attempt this. Instead, they prescribed the vertical distribution of water vapor using two alternative methods. The first is fixed specific humidity (which they called “fixed absolute humidity”), and the second is fixed relative humidity. The vertical distributions of specific humidity and relative humidity were both prescribed from observations similar to those shown in Chapter 3. Although Manabe and Wetherald’s model does not include a water budget, more modern radiative-convective models do include explicit water budgets. In these models, water vapor is introduced by evaporation from the sea surface; convection and to a much smaller degree diffusion carry the moisture upward; and precipitation removes it. An example will be given later.

Manabe and Wetherald also prescribed the vertical distribution of radiatively-active cloudiness.

The vertical distribution of ozone is important for the radiation calculation, because it is the absorption of solar radiation by ozone that accounts for the upward increase of temperature in the stratosphere; without ozone, there would be no stratosphere. Manabe and Wetherald did not attempt to compute the vertical distribution of ozone; instead they prescribed it according to observations.

From the vertical profiles of temperature, water vapor, ozone, and cloudiness, it is possible to determine the solar and terrestrial radiative energy fluxes and the net radiative cooling of the atmosphere at each level. We can then determine the net atmospheric radiative cooling, \( \mathcal{A} \), which is given by

\[
\mathcal{A} = (R_T - R_S) - (S_T - S_S).
\]

(37)

So far, so good.

How can we determine \( (F_h)_S \)? In principle, we could use the bulk aerodynamic formula, but there are problems with that approach:

- Over the ocean, the surface latent heat flux, which contributes to \( (F_h)_S \), depends on the low-level wind speed. This could be specified, but we prefer to specify as few things as possible, so this option is not attractive.
- Over land, the surface latent heat flux also depends on the soil moisture content and vegetation cover, among other things. This is more complexity than we are willing to include in the model.

To avoid such complications, we can assume that on each time step the net radiative cooling of the atmosphere is equal to the net radiative warming of the surface, i.e.
\[ ARC = S_s - R_s. \]  

(38)

From (33) and (37) it is clear that (38) must hold in equilibrium, but purely for computational reasons we assume that it also applies during the approach to equilibrium. From (36), (36), and (38), it follows that

\[ \left( F_s \right)_s = ARC. \]  

(39)

This means that the surface moist static energy flux is whatever it takes to balance the radiative cooling of the atmospheric column. Again, this is required in equilibrium, but there is no physical reason why it has to be true during the approach to equilibrium. Eq. (39) means that the atmospheric column as a whole is in energy balance, even though imbalances occur at particular levels inside the column during the approach to equilibrium.

An updated surface temperature can then be determined by time-stepping (32) with (35).

---

Figure 6.3: Flow chart for the numerical time integration in the radiative-convective equilibrium study of Manabe and Wetherald (1967).

A radiative-convective equilibrium state can be found through the procedure summarized in Fig. 6.3. Initial conditions are given for the vertical profile of atmospheric temperature. After the temperature profile has been modified by radiation, on each time step, the resulting temperature profile is checked for convective stability. Manabe and Wetherald assumed that moist convective instability exists if the temperature decreases upward more rapidly than 6.5 K km\(^{-1}\). If instability is found, the temperature profile is adjusted so as to restore a lapse rate of 6.5 K km\(^{-1}\). For the case of fixed relative humidity, the vertical distribution of water vapor is then corrected to take into account the temperature change.
The assumption that the lapse rate “adjusts” to 6.5 K km$^{-1}$ is based on the physical hypothesis that convection acts to prevent the lapse rate from becoming much steeper than the moist adiabatic lapse rate, which is close to 6.5 K km$^{-1}$ in the tropical lower troposphere (see Fig. 6.1). The physical meaning of this important hypothesis is discussed further later in this chapter.

Fig. 6.4 shows Manabe and Wetherald’s results for three cases:

- pure radiative equilibrium of the clear atmosphere with a given distribution of relative humidity,
- radiative equilibrium of the clear atmosphere with a given distribution of absolute humidity, and
- radiative-convective equilibrium of the atmosphere with a given distribution of relative humidity.

Pure radiative equilibria exhibit a troposphere and a stratosphere, with the tropopause at a fairly realistic height. An unrealistic aspect of both of the radiative equilibria is that the lower troposphere is convectively unstable, even for dry convection. In the radiative-convective calculations, this instability is assumed to be removed by convection. The radiative-convective equilibrium with fixed relative humidity is amazingly realistic, considering that the model ignores all large-scale dynamical processes.
We now present the results of more modern radiative-convective equilibrium calculations performed using the physical parameterizations of the Colorado State University general circulation model. In these simulations, the surface temperature was prescribed according to the observed zonal annual means, and annually averaged insolation was used at each latitude. The observed zonal annual means, and annually averaged insolation was used at each latitude. The

\[ \text{Figure 6.5: Temperature profiles for radiative-convective equilibrium, as simulated by the physical parameterization of the CSU GCM. The surface temperature was prescribed as a more or less realistic function of latitude, and for each latitude the annual mean insolation was used.} \]
model incorporates an elaborate theory of the interactions of cumulus convection with the large-scale circulation (see the next subsection), based on the work of Arakawa and Schubert (1974, discussed in the next sub-section), Randall and Pan (1993), and Pan and Randall (1998) as well as representations of stratiform cloud processes (Fowler et al., 1996) and boundary-layer turbulence (Suarez et al. 1983). A surface wind speed is needed to determine the surface fluxes of sensible and latent heat using bulk aerodynamic formulae similar to those discussed earlier; a value of 5 m s\(^{-1}\) was assumed for this purpose. The boundary layer, cumulus, and stratiform cloud parameterizations together determine the equilibrium distribution of moisture. The boundary layer, cumulus, stratiform cloud, and radiation parameterizations determine the distribution of temperature. Clouds in the radiative sense were neglected. The top of the model was placed at 1 mb, and 29 levels were used. Stratospheric ozone amounts were prescribed from observations. Note that in this model the distribution of moisture is predicted; this is a key difference from the work of Manabe and colleagues, discussed above.

Fig. 6.5 shows the results with and without prescribed stratospheric ozone. When ozone is present, the model produces a very obvious stratosphere in which the temperature increases upward; when ozone is neglected, on the other hand, the upper regions of the model atmosphere become more or less isothermal. The structure of the tropospheric temperature sounding is only slightly altered by the effects of stratospheric ozone.

Fels (1985) reported the results of similar radiative-convective equilibria on the sphere, but his model included a representation of photochemistry, and he emphasized the structure of the stratosphere and mesosphere, which are often referred to as the “middle atmosphere.” Because the middle atmosphere has a very stable stratification, convection is not active there. The results obtained by Fels are shown in Fig. 6.6. The observed state of the atmosphere (see panel b of Fig. 6.6) resembles that predicted by the model in the summer stratosphere, but the model is much too cold in the winter polar stratosphere. Also, the mesosphere of the real world is warm near the winter pole and cold near the summer pole, while the model predicts just the opposite. The differences between the observations and the model results can be attributed to the effects of large-scale motions, which are neglected in the model. Obviously, the motions make quite a difference in the winter middle atmosphere, where they must transport energy poleward in order to account for the differences between the observations and the results of the radiative-convective model. It appears that large-scale motions have little effect on the thermal structure of the middle atmosphere in summer, however. This indicates that in the summer the middle atmosphere is close to a state of radiative equilibrium.
The observed vertical structure of the atmosphere, and the mechanisms of vertical energy transport

Fig. 6.7 shows the observed vertical structure of the atmosphere in the tropics, the subtropical tradewind regime, and the subtropical marine stratocumulus regime. The quantities plotted are the dry static energy, defined by

\[ s \equiv c_p T + gz, \]

(40)

the moist static energy, defined by

\[ h \equiv s + Lq_v, \]

(41)

and the saturation moist static energy, defined by

\[ h^* \equiv s + Lq_v^*. \]

(42)

Here \( q_v^* \) is the saturation mixing ratio. At cold temperatures, these three quantities are nearly equal, because both the water vapor mixing ratio and the saturation mixing ratio are small. Quite generally, we have

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An Introduction to the General Circulation of the Atmosphere

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Figure 6.6: (a) Zonal mean temperatures for 15 January calculated by using a time-marched radiative-convective-photochemical model. (b) Zonal mean temperatures for January. From Fels (1985).
Figure 6.7: Representative observed soundings for Darwin, Australia, Porto Santo Island in the Atlantic tradewind regime, and San Nicolas Island, in the subtropical marine stratocumulus regime off the coast of southern California. The curves plotted show the dry static energy, the moist static energy, and the saturation moist static energy. The panels on the right cover both the troposphere and the lower stratosphere, while those on the left zoom in on the lower troposphere to show more detail. Values are divided by \( c_p \) to give units in K.
\[ s \leq h \leq h^* \, . \] (43)

In cloudy air, \( h = h^* \). In very dry air, \( h \equiv s \). In very cold air, \( h^* \equiv h \equiv s \). It can be shown that the dry static energy increases upward in a statically stable atmosphere. The dry static energy is approximately conserved under dry adiabatic processes, while the moist static energy is approximately conserved under dry adiabatic, moist adiabatic, and pseudo-adiabatic processes. The saturation moist static energy is not a conservative variable, and despite its name it does not actually carry any information about the moisture field; the sounding of the saturation moist static energy is essentially determined by the temperature sounding.

If a parcel of air containing vapor is lifted adiabatically from near the surface, it will eventually become saturated due to the cooling caused by adiabatic expansion. Prior to reaching its lifting condensation level, both the dry static energy and the moist static energy of the parcel will be conserved. Suppose that, once the lifting-condensation level has been exceeded, the moist static energy of the now-cloudy parcel remains constant, i.e., that

\[ \frac{\partial h_c}{\partial z} = 0 \, . \] (44)

This will be the case if the effects of turbulent mixing and radiation are negligible. The dry static energy of the cloudy air will increase upward due to latent heat release. The liquid water mixing ratio will increase, and the water vapor mixing ratio will correspondingly decrease. Under moist adiabatic processes, the total mixing ratio, \( q_v + l \), will be conserved.

Note that conservation of \( h \) and \( q_v + l \) implies that

\[ s_l \equiv h - L(q_v + l) \] (45)

is also conserved. We refer to \( s_l \) as the “liquid water static energy.” It is conserved under moist adiabatic processes. Precipitation from a parcel is not a moist adiabatic process, because it involves the removal of mass from the parcel. Precipitation can change the value of \( s_l \), but it does not change the value of \( h \). This means that \( h \) is “more conservative” than \( s_l \). Both variables are useful.

The temperature difference between the cloudy air and its environment at the same level is simply proportional to the saturation moist static energy difference, i.e.

\[ c_p (T_c - T) = \frac{h^*_c - h^*}{1 + \gamma} \, . \] (46)
where \( \gamma \equiv \frac{L}{c_p} \left( \frac{\partial q^*}{\partial T} \right)_p \), a subscript \( c \) denotes the cloudy air, and an overbar denotes the environment. Because the cloudy air is saturated, however, we can write

\[
c_p(T_c - \bar{T}) = \frac{h_c - h^*}{1 + \gamma}.
\]

(47)

This shows that the buoyancy of the cloudy air, as measured by the difference between its temperature and the temperature of the environment, is proportional to the difference between the moist static energy of the cloudy air and the saturation moist static energy of the environment. Recall, however, that we are considering a parcel that is lifted adiabatically from near the surface, conserving its moist static energy. This means that \( h_c \) is equal to the low-level moist static energy of the sounding. The cloudy updraft will stop when it encounters a level where \( h_c = h^* \); if this level is high and cold, then \( h^* \approx h_\text{boundary layer} \), and so we expect to find

\[
h_\text{topopause} \approx h_\text{boundary layer} \quad \text{in regions where deep convection is active.}
\]

(48)

In the terminology of Neelin and Held (1987), Eq. (48) means that the gross moist stability is small. See (29).

Suppose that the air in the cloud is neutrally buoyant with respect to its environment at each level. From (37) and (44), this implies that

\[
\frac{\partial h^*}{\partial z} = 0,
\]

(49)

which is the condition for neutral stability in saturated air, derived earlier. Note, however, that we have derived this condition without assuming that the air is saturated; it applies even when the relative humidity of the environment of the cloud is less than 100%. Eq. (49) is the condition for neutral stability with respect to a non-entraining cumulus cloud. This will be discussed in more detail later.

These ideas can be applied to the tropical (i.e. Darwin) sounding shown in Fig. 6.7. Near the surface, \( h^* > h \), indicating that the air is unsaturated. If we lift a parcel adiabatically from near the surface, its moist static energy will follow a straight, vertical line in the diagram, starting from a near-surface value. At the same time, the environmental saturation moist static energy decreases upward. After rising a kilometer or so, the vertical line representing the moist static energy traced out by adiabatic parcel ascent from the surface will be to the right of the observed sounding of saturation moist static energy, so that \( h_c > h^* \). According to (47), the parcel will then...
be positively buoyant, if it is saturated. We can thus tell, simply by looking at Fig. 6.7, that the Darwin sounding is conditionally unstable. The positive buoyancy of the lifted parcel will continue upward until the vertical line representing constant moist static energy again crosses over to the left side of the curve representing the environmental saturation moist static energy. For the Darwin soundings, this occurs at about the 15 km level, near the tropopause. We thus expect deep cumulus convection to occur in this sounding, although the mere existence of a conditionally unstable sounding is not enough, in itself, to show that cumulus convection will be

Figure 6.8: Observed locations of low-level stratus clouds, including the one west of California, as observed by ISCCP, the International Satellite Cloud Climatology Project. Cloud amounts higher than 26% are shaded.

significantly active.

An Introduction to the General Circulation of the Atmosphere
Another interesting aspect of the Darwin sounding is that the saturation moist static energy is nearly uniform with height throughout most of the troposphere. Recall [see (16)] that in an atmosphere that is neutrally stable with respect to saturated moist convection, $\frac{\partial \tilde{h}^*}{\partial z} = 0$. This may suggest that the Darwin sounding is close to neutrally stable for moist convection, and in fact this is true in a sense that will be explained later.

The subtropical “tradewind” sounding is also conditionally unstable, but only through a shallow layer. The tradewind convective layer is capped by a very strong temperature inversion, and the water-vapor mixing ratio decreases strongly upward through this “trade inversion.” The middle troposphere is much drier in the tradewind sounding than in the tropical sounding.

The subtropical marine stratocumulus sounding is not conditionally unstable at all. The sounding shows evidence of cloudiness in the lowest kilometer. The cloud layer is capped by a very strong inversion, essentially similar to the trade inversion, but residing at a lower level. Marine stratocumulus regimes occur in several places around the world, typically in association with subtropical highs. See Fig. 6.8.

The physical picture represented by the three soundings shown in Fig. 6.7 is summarized in Fig. 6.9. The subtropical marine stratocumulus regime is shown on the right side of the figure, in a region of large-scale subsidence. The sea surface temperatures are relatively cool in such regimes. Towards the left, we enter the tradewind cumulus regime, which has weaker subsidence and warmer sea surface temperatures. Finally on the left side of the figure we reach the region of deep convection, characterized by warm sea surface temperatures and large-scale rising motion.
As the air descends in the subtropical branches of the Hadley cells, it is gradually cooled by radiation. As a result, the potential temperature of the air in the subtropical free atmosphere decreases downward, or in other words it increases upward.

The lapse rate of the deep convective zones is essentially determined by convection. As already noted, however, the horizontal temperature gradients are weak throughout the tropics and subtropics, for reasons discussed by Charney (1963). This means that the lapse rate in the subtropics, above the trade inversion, must be nearly the same as the lapse rate in the tropics. We can write a rough thermodynamic balance for the descending branch of the Hadley cell as follows:

$$\omega \frac{\partial s}{\partial p} = Q_r.$$  \hspace{1cm} (50)

Here $\omega$ is the positive large-scale pressure velocity, corresponding to sinking motion; $\frac{\partial s}{\partial p}$ is the rate of change of the dry static energy with height, which we have just argued is essentially imposed, above the trade inversion, by the moist convective processes of the deep tropics; and $Q_r < 0$ is the radiative cooling. We see from (50) that the speed of the large-scale sinking motion in the subtropics is essentially determined by the requirement of thermodynamic balance. Typical clear-sky tropospheric radiative cooling rates in the tropics are on the order of 2 K per day (Fig. 6.10).

The sinking air passes through the trade inversion. How does this happen? It is remarkable, for example, that the average mixing ratio of the air suddenly increases from perhaps 1 g kg$^{-1}$ above the trade inversion to 6 or 7 g kg$^{-1}$ below the trade inversion. This is the same air, after all; how does it suddenly become so moist? The answer is that the convective vertical motions associated with the shallow stratocumulus and/or trade cumulus clouds transport moisture upwards, and deposit it at the base of the inversion, where it is used to moisten the sinking air. The moisture used for this purpose is carried up from the sea surface by a combination of turbulence and shallow cloudy convection.

The air also cools as it descends through the inversion. This cooling is produced by a combination of concentrated radiative cooling near cloud tops, evaporative cooling due to the evaporation of liquid deposited at the trade inversion level by the shallow clouds, and a downward flux of sensible heat that cools the air as it crosses the inversion.

A macroscopic view of this entrainment process is as follows. Let $A$ be an arbitrary scalar, satisfying a conservation equation that can be written in “flux form” as

$$\frac{\partial (\rho A)}{\partial t} + \nabla \cdot (\rho \mathbf{V} A) + \frac{\partial (\rho w A)}{\partial z} = -\frac{\partial F_A}{\partial z} + S_A,$$  \hspace{1cm} (51)
where $F_A \equiv \rho \overline{w'} A'$ is the upward turbulent flux of $A$, bars are omitted on the mean quantities, $S_A$

and is a source or sink of $A$, per unit volume. Integrating (51) from just below to just above the inversion, and using Leibniz’ rule, we get

$$
\frac{\partial}{\partial t} \left( \int_{z_B}^{z_B+\varepsilon} \rho A dz \right) - \Delta \rho \overline{A} \frac{\partial z_B}{\partial t} + \nabla \cdot \left( \int_{z_B}^{z_B+\varepsilon} \rho V A dz \right) - \Delta \rho \overline{V} \cdot \nabla z_B + \Delta \rho w A
$$

$$
= - \left( F_A \right)_{B+} + \left( F_A \right)_B + \int_{z_B - \varepsilon}^{z_B + \varepsilon} S_A dz \tag{52}
$$

where the indicated terms drop out as the domain of integration shrinks to zero and/or because all of the turbulence variables go to zero above the inversion. Here we have used the notation $\Delta \rho \overline{A} \equiv \left( \rho A \right)_{z_B + \varepsilon} - \left( \rho A \right)_{z_B - \varepsilon} \equiv \left( \rho A \right)_{B+} - \left( \rho A \right)_B$, and henceforth subscripts $B+$ and $B$ denote levels just above and just below the inversion, respectively. For $A \equiv 1$, (52) reduces to mass conservation in the form

Figure 6.10: Estimates of the clear-sky radiative cooling rate in the eastern and central Pacific, based on ECMWF data.
\[ \rho_{B^+} \left( \frac{\partial z_B}{\partial t} + \mathbf{V}_{B^+} \cdot \nabla z_B - w_{B^+} \right) = \rho_{B} \left( \frac{\partial z_B}{\partial t} + \mathbf{V}_B \cdot \nabla z_B - w_B \right) \equiv E, \]

(53)

where \( E \) is the downward mass flux across the inversion. In essence, (53) simply says that the mass flux is continuous across the PBL top, i.e. no mass is created or destroyed between levels \( B \) and \( B^+ \). We interpret \( E \) as the mass flux due to the turbulent entrainment of free atmospheric air across the inversion. With the definition of \( E \) as given by (53), we can simplify (52) to

\[ -AE \left( F_A \right)_B + \int_{z_a-z^e}^{z_a+z^e} S_A\,dz \]

(54)

For \( S_A = 0 \), (54) simply says that the total flux of \( A \) must be continuous across the inversion. Notice that for \( \Delta A \neq 0 \), a mass flux across the inversion is generally associated with the convergence of a turbulent flux of \( A \) at level \( B \). This flux convergence changes the \( A \) of entering particles from \( AB^+ \) to \( AB \). Lilly (1968; Fig. 6.11) was the first to derive (54) using the approach followed above.

Figure 6.11: Douglas Lilly, who did important work on a wide range of topics, including cumulus convection, numerical methods, gravity waves, and stratocumulus clouds.

As a simple example, consider the moistening of the air as it moves down across the inversion. The dry entrained air is moistened by an upward moisture flux that converges “discontinuously” at level \( B \). This is described by

\[ -\Delta q_{T}E = \left( F_{q_{T}} \right)_B, \]

(55)

which is a special case of (53). Here \( q_T \equiv q_v + l \) is the total water mixing ratio.
After sinking through the trade inversion, the air is subjected to friction. This causes its angular momentum to decrease. As it flows back equatorward, near-surface easterlies result.

Figure 6.12: Representative observed soundings for Denver, Colorado, and Barrow, Alaska, for July and January. The curves plotted show the dry static energy, the moist static energy, and the saturation moist static energy. The panels on the right cover both the troposphere and lower stratosphere, while those on the right zoom in on the lower troposphere to show more detail. Values are divided by $c_p$ to give units in K.

An Introduction to the General Circulation of the Atmosphere
Turning now to middle and high latitudes, Fig. 6.12 shows representative summer and winter soundings for Denver, Colorado and Barrow, Alaska. The Denver sounding is conditionally unstable in summer, but the dry near-surface air has to be lifted quite a long way before it can become positively buoyant; we expect to see high cloud bases. The winter sounding for Denver is strongly stable near the surface. The Barrow sounding is quite stable all year, but especially so in winter. Note that the tropopause is much lower at Barrow that at Denver. In summer, there are low clouds at Barrow.

The convective mass flux

The preceding discussion suggests that moist convection is important in (at least) two ways:

- As shown observationally by Riehl and Malkus (1958), moist convection is the primary mechanism to transport energy upward in the deep tropics. Convection also transports moisture and momentum, as well as various chemical constituents.

- As hypothesized by Manabe and Wetherald (1967), convection acts to prevent the lapse rate in convectively active regions from exceeding a value close to the moist adiabatic lapse rate. Even if we accept the validity of this hypothesis, we are faced with the problem of determining when and where convection is active.

In addition, of course, moist convection is important because:

- Convection produces a large fraction of the Earth’s precipitation.

- Convection generates radiatively important stratiform clouds, especially in the upper troposphere in regions of deep convection.

For the four reasons summarized in the bullets above, moist convection is crucially important for the general circulation of the Earth’s atmosphere. It is no exaggeration to say that we cannot understand the general circulation unless we understand the interactions between the general circulation and moist convection. In fact there is an enormous literature on this subject of “cumulus parameterization,” and many contentious issues remain unresolved. Additional issues have not even been confronted yet. Current ideas are that cumulus convection exerts its effects on the large-scale stratification by transporting mass vertically (the “cumulus mass flux”), and that the intensity of convection is regulated by the processes that act to produce convective instability; these include radiative cooling of the air relative to the temperature of the lower boundary, surface fluxes of sensible and latent heat, and the effects of both horizontal and vertical advection, including moisture convergence from neighboring columns.
The effects of convection on the large-scale state can be analyzed following Arakawa and Schubert (1974; hereafter AS; see Fig. 6.13). Let an overbar denote a suitable average. The averaged budget equations for mass, dry static energy, water vapor mixing ratio, and liquid water mixing ratio are:

\[ 0 = -\nabla \cdot \left( \rho \vec{V} \right) - \frac{\partial (\rho \vec{w})}{\partial z}, \]

(56)

\[ \rho \frac{\partial \overline{s}}{\partial t} = -\rho \nabla \cdot \overline{\vec{s}} - \rho \overline{\vec{w}} \frac{\partial \overline{s}}{\partial z} + \overline{Q_r} + \rho L \overline{C} - \frac{\partial F_s}{\partial z}, \]

(57)

\[ \rho \frac{\partial \overline{q_v}}{\partial t} = -\rho \nabla \cdot \overline{\vec{q}_v} - \rho \overline{\vec{w}} \frac{\partial \overline{q_v}}{\partial z} - \rho \overline{C} - \frac{\partial F_{q_v}}{\partial z}, \]

(58)

\[ \rho \frac{\partial \overline{l}}{\partial t} = -\rho \nabla \cdot \overline{\vec{l}} - \rho \overline{\vec{w}} \frac{\partial \overline{l}}{\partial z} + \rho \overline{C} - \frac{\partial F_l}{\partial z} - \overline{\chi}. \]

(59)

Here \( \rho \) is the density of the air, which is presumed to be quasi-constant at each height; \( s \equiv c_p T + g z \) is the dry static energy; \( q \) is the water vapor mixing ratio; \( w \) is the vertical velocity;
\( \nabla \) is the horizontal velocity; and \( \chi \) is the rate at which liquid water is being converted into precipitation, which then falls out and so acts as a sink of \( I \). The vertical “eddy fluxes,” \( F_v \equiv \rho w_s - \rho \overline{w_s} \) and \( F_q \equiv \rho w_q - \rho \overline{w_q} \), can in principle represent quite a variety of physical processes, but here we assume for simplicity that above the boundary layer these fluxes are due only to the vertical currents associated with cumulus convection.

AS used a very simple cumulus cloud model to formulate the eddy fluxes that appear in (57) - (59) in terms of a convective mass flux and the differences between the in-cloud and environmental soundings. The cloud model was also used to formulate the net condensation rate, \( C \), per unit mass flux. AS allowed the possibility that clouds of many different “types” coexist; here a cloud type can be roughly interpreted as a cloud size category. We now briefly explain the AS parameterization, using a single cloud type for simplicity.

As a first step, we divide the domain into an arbitrary number \( N \) of subdomains, each having a characteristic fractional area \( \sigma_i \), a characteristic vertical velocity \( w_i \), and corresponding characteristic values of the moist static energy, dry static energy, water vapor mixing ratio, and all of the other variables of interest. Some of the subdomains represent cloudy updrafts or downdrafts, while others could represent mesoscale subdomains or the broad “environment” of the clouds. The fractional areas must sum to unity:

\[
\sum_{i=1}^{N} \sigma_i = 1. \tag{60}
\]

The area-averaged vertical velocity and moist static energy satisfy:

\[
\sum_{i=1}^{N} \sigma_i w_i = \overline{w}, \tag{61}
\]

\[
\sum_{i=1}^{N} \sigma_i h_i = \overline{h}, \tag{62}
\]

and other area-averages are constructed in a similar way. It then follows that

\[
F_h = \rho \overline{w h} - \rho \overline{w} \overline{h} = \sum_{i=1}^{N} M_i \left( h_i - \overline{h} \right), \tag{63}
\]

where
\[ M_i \equiv \rho \sigma_i (w_i - \bar{w}) \]  

(64)

is the “convective mass flux” associated with cloud type \( i \). The convective mass flux is a key concept. It represents the rate at which mass is pumped through the convective circulations -- through the updraft, through the compensating sinking motion outside the updraft, and through the horizontal branches of the convective circulation that connect the updraft and the sinking motion.

We now assume for simplicity that the effects of mesoscale organization and convective-scale downdrafts can be neglected, so that the cloudy layer consists of concentrated convective updrafts of various sizes and intensities, embedded in a broad uniform environment. We use a superscript tilde to denote an environmental value, and a subscript to denote the collective properties of the cloudy updrafts. Then (60) simplifies to

\[ \sigma_c + \bar{\sigma} = 1 , \]

(65)

where

\[ \sigma_c \equiv \sum_{\text{all clouds}} \sigma_j \]

(66)

is the total fractional area covered by all of the convective updrafts, and \( \bar{\sigma} \) is the fractional area of the environment. Similarly, (61) and (62) become

\[ \sigma_c w_c + \bar{\sigma} \bar{w} = \bar{w} , \]

(67)

\[ \sigma_c h_c + \bar{\sigma} \bar{h} = \bar{h} , \]

(68)

where we define

\[ w_c \equiv \sum_{\text{all clouds}} \frac{\sigma_j w_j}{\sigma_c} , \]

(69)

\[ h_c \equiv \sum_{\text{all clouds}} \frac{\sigma_j h_j}{\sigma_c} . \]

(70)
Why $\sigma_c$ is small, and how this simplifies things

It is observed that

$$\sigma_c \ll 1 \quad \text{and} \quad \tilde{\sigma} \equiv 1.$$  \hfill (71)

This means that the cumulus updrafts occupy only a very small fraction of the area. A simple explanation for this important fact was given by Bjerknes (1938). Suppose that at a certain time the temperature is horizontally uniform, with lapse rate

$$\Gamma \equiv -\frac{\partial T}{\partial z}.$$  \hfill (72)

Consider temperature changes due to adiabatic vertical motion only. As discussed earlier, in a cloudy region, the temperature satisfies

$$\frac{\partial T_c}{\partial t} = w_c (\Gamma - \Gamma_m),$$  \hfill (73)

while in a neighboring clear region,

$$\frac{\partial T}{\partial t} = \tilde{w} (\Gamma - \Gamma_d).$$  \hfill (74)

If the sounding is conditionally unstable, then

$$\Gamma - \Gamma_m > 0 \quad \text{and} \quad \Gamma - \Gamma_d < 0.$$  \hfill (75)

Suppose that the cloudy air is rising and the environmental air is sinking, so that $w_c > 0$, $\tilde{w} < 0$, and $w_c - \tilde{w} > 0$. Comparing (75) with (73) and (74), we see that when the sounding is conditionally unstable both $T_c$ and $\tilde{T}$ will tend to increase with time. The buoyancy of the updraft is proportional to $T_c - \tilde{T}$. An increase in $T_c$ favors convection because it will tend to increase the buoyancy of the cloudy air, but an increase in $\tilde{T}$ will tend to decrease the buoyancy. Which effect wins out? The answer depends on the value of $\sigma_c$.

The mean vertical motion satisfies (67), from which it follows that

$$w_c = \tilde{w} + (1 - \sigma_c)(w_c - \tilde{w}),$$  \hfill (76)
\[ \tilde{w} = \bar{w} - \sigma_c (w_c - \tilde{w}). \]  
(77)

Subtracting (74) from (73), and substituting from (76) and (77), we find that
\[
\frac{\partial}{\partial t}(T_c - \bar{T}) = w_c (\Gamma - \Gamma_m) - \bar{w} (\Gamma - \Gamma_d)
= \bar{w} (\Gamma_d - \Gamma_m) + (w_c - \tilde{w}) \left[ (1 - \sigma_c) (\Gamma - \Gamma_m) + \sigma_c (\Gamma - \Gamma_d) \right].
\]  
(78)

We focus on the quantity \[ \left[ (1 - \sigma_c) (\Gamma - \Gamma_m) + \sigma_c (\Gamma - \Gamma_d) \right], \] which multiplies \( w_c - \tilde{w} \) on the second line of (78). Inspection of (78) shows that \[ \left[ (1 - \sigma_c) (\Gamma - \Gamma_m) + \sigma_c (\Gamma - \Gamma_d) \right] \] is maximized for \( \sigma_c \to 0 \). The physical interpretation is simple. With a conditionally unstable sounding, saturated rising motion is aided by positive buoyancy created through condensation, while unsaturated sinking motion must fight against the dry-stable stratification. The rate of temperature increase in the updraft is proportional to the updraft speed, while the rate of temperature increase in the downdraft is proportional to the downdraft speed. Therefore, the convection is favored by rapid rising motion in the cloudy region, and slow sinking motion in the clear region, both of which can be achieved, for a given value of \( w_c - \tilde{w} \), by making the updraft narrow, and the downdraft broad.

With this simple theory, Bjerknes (1938) explained the observed smallness of \( \sigma_c \).

The smallness of \( \sigma_c \) is crucially important for the general circulation because it means that even in the regions where deep convection is most active there is a lot of clear sky where no phase changes occur. The convective clouds “tunnel through” deep layers of (mostly) unsaturated air. Cloud processes would be relatively simple if they involved only uniform cloudiness over large regions. The importance of narrow saturated updrafts in clear environments makes the interaction of moist convection with the general circulation a much more subtle (and interesting) problem than it would otherwise be.

In view of (61), we can write (68) as
\[ \tilde{h} \equiv \bar{h}, \]  
(79)

It is not true, however, that \( \tilde{w} \equiv \bar{w} \), because the cumulus updrafts are typically several orders of magnitude stronger than the large-scale vertical motions, i.e.,
\[ w_i \gg \bar{w}. \]  
(80)
Similarly, it is *not* true in general that \( \bar{T} \equiv \bar{t} \), because there may be no liquid water at all in the environment of the convective clouds.

Using (80), we can approximate (64) by
\[
M_i \equiv \rho \sigma_i w_i .
\] (81)

From this point we simplify the discussion by considering only one type of convective cloud, whose properties are denoted by subscript \( c \). We write
\[
F_s \equiv M_c \left( s_c - \bar{s} \right) ,
\] (82)
\[
F_{qv} \equiv M_c \left[ (q_v)_c - \bar{q}_v \right] ,
\] (83)
\[
F_l \equiv M_c \left( l_c - \bar{l} \right) .
\] (84)

Here \( s_c , (q_v)_c , \) and \( l_c \) are the in-cloud dry static energy, water vapor mixing ratio, and liquid water mixing ratio, respectively, and
\[
M_c \equiv \rho \sigma_c w_c .
\] (85)

Note that, in (84), we allow the possibility of liquid water in the environment of the cumulus clouds, but we do *not* assume that \( \bar{l} \equiv \bar{t} \), because it is quite possible that the only liquid present is in the convective updrafts. We also use
\[
\bar{s} = (1 - \sigma_c) \bar{s} + \sigma_c s_c ,
\] (86)
\[
\bar{q}_v = (1 - \sigma_c) \bar{q}_v + \sigma_c (q_v)_c ,
\] (87)
\[
\bar{l} = (1 - \sigma_c) \bar{l} + \sigma_c l_c ,
\] (88)
\[
\bar{C} = (1 - \sigma_c) \bar{C} + \sigma_c C_c ,
\] (89)
\[ \bar{\chi} = (1 - \sigma_c) \tilde{\chi} + \sigma_c \chi_c. \]  

(90)

We can now rewrite (57) - (59) as

\[
\rho \frac{\partial s}{\partial t} = -\rho \nabla \cdot \nabla \tilde{s} - \rho \tilde{w} \frac{\partial \tilde{s}}{\partial z} + Q_r + \rho L (\tilde{C} + \sigma_c C_c) - \frac{\partial}{\partial z} \left[ M_c (s_c - \tilde{s}) \right],
\]

(91)

\[
\rho \frac{\partial \tilde{q}_v}{\partial t} = -\rho \nabla \cdot \tilde{q}_v - \rho \tilde{w} \frac{\partial \tilde{q}_v}{\partial z} - \rho (\tilde{C} + \sigma_c C_c) - \frac{\partial}{\partial z} \left[ M_c \left( (q_v)_c - \tilde{q}_v \right) \right],
\]

(92)

\[
\rho \frac{\partial \tilde{l}}{\partial t} = -\rho \nabla \cdot \tilde{l} - \rho \tilde{w} \frac{\partial \tilde{l}}{\partial z} + \rho (\tilde{C} + \sigma_c C_c) - \frac{\partial}{\partial z} \left[ M_c \left( (l_c)_c - \tilde{l} \right) \right] - \left[ (1 - \sigma_c) \tilde{\chi} + \sigma_c \chi_c \right].
\]

(93)

The convective condensation rate, \( C_c \), appears in all three of these equations, as would be expected.

**A simple cumulus cloud model**

To go further, we need to describe what is going on inside the convective updrafts; for example, we need to know the soundings in side the updrafts. A simple cumulus cloud model is needed. We assume that all cumulus clouds originate from the top of the PBL, carrying the mixed-layer properties upward. The mass flux changes with height according to

\[
\frac{\partial M_c(z)}{\partial z} = E(z) - D(z).
\]

(94)

Here \( E \) is the entrainment rate, and \( D \) is the detrainment rate. The in-cloud profile of moist static energy, \( h_c(z) \), is governed by

\[
\frac{\partial}{\partial z} \left[ M_c(z) h_c(z) \right] = E(z) \tilde{h}(z) - D(z) h_c(z) = E(z) \tilde{h}(z) - D(z) h_c(z).
\]

(95)

Here we have used (79). There are no source or sink terms in (95) because the moist static energy is unaffected by phase changes and/or precipitation processes, and we neglect radiative effects. By combining (94) and (95), we can show that
\[
\frac{\partial h_c(z)}{\partial z} = \frac{E(z)}{M_c} \left[ \bar{h}(z) - h_c(z) \right].
\]

(96)

This means that \( h_c(z) \) is affected by entrainment, which dilutes the cloud with environmental air, but not by detrainment, which has been assumed to expel from the cloud air that has the average moist static energy of the cloud at each level.

Similarly, we can write

\[
\frac{\partial}{\partial z} \left( M_c s_c \right) = E \bar{s} - D s_c + \rho \sigma_c L C_c,
\]

(97)

\[
\frac{\partial}{\partial z} \left[ M_c (q_v)_c \right] = E \bar{q}_v - D (q_v)_c - \rho \sigma C_c,
\]

(98)

\[
\frac{\partial}{\partial z} \left( M_c l_c \right) = E \bar{l} - D l_c + \rho \sigma C_c - \chi_c.
\]

(99)

In (99), we do not assume that \( \bar{I} = \bar{L} \).

A simple microphysical model is needed to determine \( \chi_c \), i.e. to determine how much of the condensed water is converted to precipitation, and the fate of the precipitation. The role of convectively generated precipitation, which drives convective downdrafts and moistens the lower troposphere by evaporating as it falls, is actually an important issue, but it will not be discussed here.

“Compensating subsidence”

By using (97)-(99), the large-scale budget equations can be rewritten in a very interesting way, as follows. First, consider the dry static energy. We write

\[
\frac{\partial}{\partial z} \left[ M_c (s_c - \bar{s}) \right] = \frac{\partial (M_c s_c)}{\partial z} - M_c \frac{\partial \bar{s}}{\partial z} - \bar{s} \frac{\partial M_c}{\partial z}.
\]

(100)

Now substitute from (94) and (97) into (100) to obtain
\[
\frac{\partial}{\partial z} \left[ M_c (s_c - \bar{s}) \right] = (E - D) s_c + \rho L \sigma c C_c - M_c \frac{\partial \bar{s}}{\partial z} - \frac{s_c}{s}(E - D)
\]

(101)

This allows us to rewrite (91) as

\[
\rho \frac{\partial s}{\partial t} = -\rho \nabla \cdot \nabla s - \rho \bar{w} \frac{\partial \bar{s}}{\partial z} + \frac{\bar{Q}_r}{\sigma_c} + \rho L C_c + M_c \frac{\partial \bar{s}}{\partial z} - D(s_c - \bar{s})
\]

(102)

The last two terms on the right-hand side of (102) represent the cumulus effects, and the first of these in particular is quite interesting. It “looks like” an advection term. It represents the warming of the environment due to the downward advection of air from above, with higher dry static energies, by the environmental sinking motion that compensates for the rising motion in the cloudy updraft. The environmental sinking motion is called “compensating subsidence.”

The role of compensating subsidence can be seen more explicitly by combining the two “vertical advection” terms of (102), and using (77), to obtain

\[
\rho \frac{\partial \bar{s}}{\partial t} = -\rho \nabla \cdot \nabla \bar{s} - \bar{M} \frac{\partial \bar{s}}{\partial z} + \frac{\bar{Q}_r}{\sigma_c} + \rho L \bar{C} + M_c \frac{\partial \bar{s}}{\partial z} - D(s_c - \bar{s})
\]

(103)

where

\[
\bar{M} \equiv \rho \bar{w} - M_c
\]

(104)

is the environmental mass flux. The reason that \( \bar{M} \) appears in (103) is that (79) applies, i.e. \( \bar{s} = \bar{s} \). The last term on the right-hand side of (103) represents the effects of detrainment. You may be surprised to see that the cumulus condensation rate does not appear in (102) or (103). An interpretation is that condensation inside the updraft cannot directly warm the environment. Since almost the entire area is the environment, this means that condensation in the updrafts does not, to any significant degree, directly affect the area-averaged dry static energy. Instead, the effects of condensation are felt indirectly, through the compensating subsidence term, which we have already interpreted. The physical role of condensation, then, is to make possible the convective updraft that drives the compensating subsidence, which in turn warms the environment. This is how condensation warms indirectly. Note that the vertical profile of the indirect condensation heating rate due to compensating subsidence is in general different from the vertical profile of the convective condensation rate itself.

In a similar way, we find that the water vapor budget equation can be rewritten as
\[
\rho \frac{\partial q_v}{\partial t} = -\rho \mathbf{V} \cdot \nabla q_v - M \frac{\partial q_v}{\partial z} - \rho \tilde{C} + D \left[ (q_v)_c - q_v \right],
\]

(105)

which is similar to (103). Eq. (105) describes the convective drying in terms of convectively-induced subsidence in the environment, which brings down drier air from aloft. Detrainment of water vapor from the convective clouds can moisten the environment.

Finally, similar methods can be used to write the liquid water budget equation as

\[
\rho \frac{\partial l}{\partial t} = -\rho \mathbf{V} \cdot \nabla l - \rho w \frac{\partial l}{\partial z} + \rho C_c + M_c \frac{\partial l}{\partial z} + D \left[ (l)_c - \check{l} \right] - \left[ (1 - \sigma_c) \check{\chi} + \sigma_c \chi_c \right].
\]

(106)

In (106), we cannot combine the two “vertical advection” terms, because one of them involves \( \partial l / \partial z \), while the other involves \( \partial l / \partial z \). Detrained liquid (or ice) can persist in the form of stratiform “anvil” and cirrus clouds.

Within the limits of applicability of the assumptions discussed above, (103) and (105) - (106) are equivalent to (57) - (59).

The mass flux profile

As discussed earlier [see (47)], the buoyancy of the cloudy air at height \( z \), is approximately given by

\[
B(z) \equiv T_c - \mathcal{T} \sim \frac{1}{c_p} \left[ \left( h_{\check{c}}(z) - \check{h}^*(z) \right) \right],
\]

(107)

where \( \check{h}^* \) is the saturation moist static energy. Because more rapidly entraining clouds lose their buoyancy at lower levels, in effect the cloud types differ according to their cloud-top heights, for a given sounding. The cloud top occurs at level \( \hat{p} \), where

\[
B(\hat{p}) = 0.
\]

(108)

This equation can be used to find \( \hat{p} \), after the in-cloud sounding has been determined using (95) and (97) - (99).

To determine the entrainment rate, AS assumed that

\[
E = \lambda M_c,
\]

(109)
where $\lambda$, which is called the fractional entrainment rate and has the units of inverse length, is assumed to be a constant (with height) for each cloud type. Larger values of $\lambda$ mean stronger entrainment; $\lambda = 0$ means no entrainment; $\lambda < 0$ has no physical meaning and so is not allowed. For a given sounding, clouds with smaller values of $\lambda$ (weaker fractional entrainment) will have higher tops. Because $\lambda$ is assumed to be a constant with height, the solution of (94) gives an exponential profile for $\eta(\lambda, z)$, from the cloud-base level to the cloud top level.

For simplicity, AS assumed that detrainment occurs only at cloud top. This means that the mass flux jumps discontinuously to zero at cloud top. Below the cloud-top, we have entrainment but no detrainment, so that (94) reduces to

$$\frac{\partial M_c(z)}{\partial z} = E(z).$$

(110)

Combining (109) and (110), and using the assumption that $\lambda$ is constant with height for each cloud type, we find that

$$M_c(z, \lambda) = M_b(\lambda) \exp(\lambda z),$$

(111)

where $M_b(\lambda)$ is the “cloud-base mass flux distribution function.” We define a normalized mass flux, denoted by $\eta(\lambda, z)$; the normalization is in terms of the cloud-base mass flux:

$$M_c(z, \lambda) \equiv M_b(\lambda) \eta(\lambda, z).$$

(112)

Note that by virtue of its definition, $\eta(\lambda, z_B) = 1$; here $z_B$ is the cloud-base height.

At this point, we have the equations we need to determine the rates at which cumulus convection warms and dries the large-scale state, if we can determine $M_b(\lambda)$, which is a measure of convective intensity. Since the value of $\lambda$ determines the height of cloud top for a given sounding, we can think of $M_b(\lambda)$ as being roughly equivalent to a single function of height. A physical idea is needed to determine $M_b(\lambda)$.

What determines the intensity of the convection?

To determine the intensity of convective activity, AS proposed a “quasi-equilibrium” hypothesis, according to which the convective clouds quickly convert whatever moist convective available potential energy is present in convectively active atmospheric columns into convective kinetic energy. The starting point for the quasi-equilibrium closure is the recognition that
cumulus convection occurs as a result of moist convective instability, in which the potential energy of the mean state is converted into the kinetic energy of cumulus convection.

AS defined the “cloud work function,” $A$, for a cumulus subensemble, as a vertical integral of the buoyancy of the cloud air with respect to the large-scale environment:

$$A(\lambda) = \int_{z_a}^{z_D(\lambda)} g \frac{c_p}{\overline{T}(z)} \eta(z, \lambda) \left[ s_v(z, \lambda) - \bar{s}_v(z) \right] dz.$$

(113)

Here $z_D(\lambda)$ is the height of the detrainment level for cloud type $\lambda$; and $s_v$ denotes the virtual static energy. From (113) we see that the function $A(\lambda)$ is a property of the large-scale environment. A positive value of $A(\lambda)$ means that a cloud with fractional entrainment rate can convert the potential energy of the mean state into convective kinetic energy. For $\lambda = 0$, $A(\lambda)$ is equivalent to the convective available potential energy (CAPE), as conventionally defined.

Numerical models use the conservation equations for thermodynamic energy and moisture to predict $T(z)$ and $q(z)$, from which $A(\lambda)$ can be determined; therefore, these models indirectly predict $A(\lambda)$. By taking the time derivative of (113), and using the conservation equations for thermodynamic energy and moisture, AS derived an equation that can be written in simplified form as

$$\frac{dA(\lambda)}{dt} = J(\lambda) M_B(\lambda) + F(\lambda).$$

(114)

The $JM_B$ term of (114) represents all of the terms involving convective processes, each of which turns out to be proportional to $M_B$. The $JM_B$ term actually represents an integral over cloud types, and is written here as a product merely to simplify the discussion. The quantity $J(\lambda)$ symbolically represents the kernel of the integral, which is a property of the large-scale sounding; see AS for details. The $JM_B$ term of (114) tends to reduce $A(\lambda)$, because cumulus convection stabilizes the environment, so that $J(\lambda)$ is usually negative. Keep in mind that an equation like (114) holds for each cumulus subensemble.

The $F(\lambda)$ term of (114) represents what AS called the “large-scale forcing,” i.e., the rate at which the cloud work function tends to increase with time due to a variety of processes including:

- horizontal and vertical advection by the mean flow;
• the surface turbulent fluxes of sensible and latent heat, and the rate of change of the planetary boundary-layer depth;
• radiative heating and cooling;
• precipitation and turbulence in stratiform clouds.

Note that some of these “forcing” processes, e.g. those involving boundary-layer turbulence and stratiform clouds, are themselves parameterized processes that may involve fluctuations on small spatial scales; for this reason it seems inappropriate to describe the collection of processes that contribute to $F$ as “large-scale;” a better term would be “non-convective.”

AS assumed quasi-equilibrium (QE) of the cloud work function, i.e.

$$\frac{dA(\lambda)}{dt} = JM_B(\lambda) + F(\lambda) \equiv 0 \text{ when } F(\lambda) > 0.$$  \hspace{1cm} (115)

Eq. (115) means that the moist convective instability generated by the forcing, $F(\lambda)$, is very rapidly consumed by cumulus convection, i.e. the two terms on the right-hand side of (115) approximately balance each other. In a steady-state situation, this balance is of course trivially satisfied, by definition. The physical content of (115) is, therefore, the assertion that near-balance is maintained even when $F(\lambda)$ is varying with time, provided that the variations of $F(\lambda)$ are sufficiently slow. The cumulus ensemble thus closely follows the lead of the forcing, like a defensive basketball player (the convection) playing man-to-man against an offensive player (the forcing). Keep in mind, however, that the forcing depends on the large-scale circulation, which is strongly affected by the convection, just as the play of an offensive basketball player is strongly affected by the moves of his or her defensive opponent. We should not imagine that the forcing is “given” and that the convection just meekly responds to it. The convection and the forcing evolve together according to the rules defined by the combination of large-scale dynamics and cloud dynamics.

The QE approximation is expected to hold if $\tau_{LS}$, the time scale for changes in $F(\lambda)$, is much longer than the “adjustment time,” $\tau_{adj}$, required for the convection to consume the available CAPE; this allows the convection to keep up with the changes in $F(\lambda)$. AS introduced the concept of $\tau_{adj}$ by describing what would happen if a conditionally unstable initial sounding were modified by cumulus convection, without any forcing to maintain the CAPE over time; they asserted that the CAPE would be consumed by the convection (i.e. converted into convective kinetic energy) on a time-scale that they defined as and estimated to be on the order of a couple of hours. Just such an unforced convective situation has been numerically simulated by Soong and Tao (1980) and others, using high-resolution cloud models; their results are consistent with the scenario of AS. If the adjustment time is on the order of $10^3$ to $10^4$ s, then use...
of (116) is justified, as an approximation, for the simulation of “weather” whose time scale is on the order of one day or longer, i.e., at least one order of magnitude longer than \( \tau_{\text{adj}} \).

By using (115), together with

\[
|JM| \sim \frac{A}{\tau_{\text{adj}}},
\]

(116)

AS found that

\[
A \sim \tau_{\text{adj}}F \ll \tau_{LS}F,
\]

(117)

where \( \tau_{LS} \) is the time-scale on which the forcing itself is varying. This means that the cloud work function is “small” compared to \( \tau_{LS}F \), which is the value that the cloud work function would take if the forcing acted without opposition over a time scale \( \tau_{LS} \). Although we should expect to see day-to-day variations of \( A \), we should not expect to see values as large as \( \tau_{LS}F \). This means that \( A \) is trapped in the range of values between zero (since by definition \( A \) cannot be negative) and \( \tau_{\text{adj}}F \). In this sense, \( A \) is “close to zero” (see also Xu and Emanuel, 1989).

Based on the analysis above, it can be asserted that the cloud work function (or the CAPE) “is quasi-invariant with time,” i.e. that

\[
\frac{dA}{dt} \equiv 0,
\]

(118)

which is a short-hand form of Eq. (115); and that “the CAPE is small” in convectively active regimes, i.e.

\[
A \equiv 0,
\]

(119)

which is a short-hand form of Eq. (117). A sounding for which \( A \equiv 0 \) tends to follow a saturated moist adiabat, throughout the depth of the convective layer. This provides a rationalization for Manabe and Wetherald’s (1967) assumption that the lapse rate cannot exceed the moist adiabatic lapse rate, which they approximated as 6.5 K km\(^{-1} \).
Unfortunately, because (118) and (119) are short-hand forms, they are subject to misinterpretation. For example, data like those shown in Fig. 6.14 are sometimes viewed as being inconsistent with QE. It is natural to wonder how the CAPE be described as “quasi-invariant” when it is observed to undergo such “large” changes. This point of view appears to be based on the tacit assumption that when we say that the changes in the CAPE are “small,” we mean that they are small compared with the time average of the CAPE. In fact, however, it should be clear from the preceding discussion that this is not what is meant at all. Instead, we mean that the changes in the CAPE are small compared to those that would occur if the convection were somehow suppressed while the non-convective processes continued to increase the CAPE with time. Eq. (115) does not imply that $A$ is invariant from day to day, and the observed day-to-day changes in the CAPE, such as those shown in Fig. 6.14, are not in conflict with QE. What QE does imply is that the changes in the CAPE that we actually see, from day to day, are much smaller than they would be if the negative convective term of (115) could somehow be suppressed, so that the positive (under disturbed conditions) forcing term could have its way with the sounding.

A practical application of (115) is to solve it for the convective mass flux as a function of cloud type, $\lambda$. After discretization this leads to a system of linear equations (Lord et al. 1982). Although the system is linear, the mass flux distribution function, $M_\theta(\lambda)$, is required to be non-negative for all $\lambda$. This cannot be guaranteed without making additional assumptions (e.g. Hack et al., 1984). An alternative approach that avoids these difficulties was proposed by Randall and Pan (1993) and Pan and Randall (1998).

For further discussions of cumulus parameterization see the collections of essays edited by Emanuel and Raymond (1993) and Smith (1998), as well as the articles by Randall et al. (2003) and Arakawa (2004).

Summary

The observed vertical structure of the atmosphere is controlled, to a remarkable degree, by diabatic processes. For example, the observed height of the tropopause is approximately that predicted by radiative-convective models, which completely ignore the effects of large-scale dynamics. Obviously this does not mean that dynamics is unimportant, but it does suggest that dynamics is strongly constrained by radiation and convection. At the same time, the heating due to radiation, convection, and boundary-layer turbulence is strongly controlled by the general circulation. It is impossible to understand the circulation without understanding the heating, and vice versa. Further discussion is given in later chapters.

Problems

1. Derive

\[ F_h \equiv \rho \overline{wh} - \rho \overline{w} \overline{h} = \sum_{i=1}^{N} M_i \left( h_i - \overline{h} \right). \]

2. Refer to Table 6.1.

a) Consider the number in the bottom right corner of the table, i.e., 1.30 x 10^{15} J s^{-1}. Assuming that there is no net radiative source or sink of moist static energy at any level in the Equatorial Trough Zone, how can you account for this number? Make a simple sketch to explain your answer.

b) Using the numbers given in the table, estimate the total upward transport of moist static energy across the 500 mb surface in the Equatorial Trough Zone, in J s^{-1}.

c) Using the numbers given in the table, estimate the upward transport of moist static energy across the 500 mb surface due to the large-scale rising motion in the Equatorial Trough Zone, in J s^{-1}.

d) Using your answers from above, work out the numerical value of \( F_h \)_{500mb}, the upward flux of moist static energy due to convection at 500 mb, in W m^{-2}. Assume that the area covered by the Equatorial Trough Zone is 4 x 10^{13} m^2.
e) Estimate a rough numerical value (in kg m\(^{-2}\) s\(^{-1}\)) of the convective mass flux, \(M_c\), at the 500 mb level in the ITCZ. You will need to use

\[
F_h \equiv M_c \left( h_c - \overline{h} \right),
\]

where \(h_c\) is the in-cloud moist static energy, and \(\overline{h}\) is the large-scale mean moist static energy. State your assumptions.

3. Suppose that moist static energy is simply conserved, i.e.

\[
\frac{\partial h}{\partial t} = -\nabla \cdot (Vh) - \frac{\partial (\omega h)}{\partial z}.
\]

The density has been omitted here and throughout the rest of this problem for simplicity. The corresponding continuity equation is

\[
0 = -\nabla \cdot V - \frac{\partial \omega}{\partial z}.
\]

a) By using Reynolds averaging, show that

\[
\frac{\partial \overline{h}}{\partial t} = -\nabla \cdot (\overline{Vh} + \overline{V'h'}) - \frac{\partial}{\partial z} \left( \overline{\omega h} + \overline{\omega'h'} \right).
\]

For large-scale averages this can be approximated by

\[
\frac{\partial \overline{h}}{\partial t} \equiv -\nabla \cdot (\overline{Vh}) - \frac{\partial}{\partial z} \left( \overline{\omega h} + \overline{\omega'h'} \right).
\]

b) Show that the moist static energy variance, \(\overline{h'^2}\), satisfies

\[
\frac{\partial \overline{h'^2}}{\partial t} = -\nabla \cdot \left( \overline{Vh'^2} + \overline{V'h'^2} \right) - \frac{\partial}{\partial z} \left( \overline{\omega h'^2} + \overline{\omega'h'^2} \right) - 2 \nabla \overline{h'} \cdot \nabla \overline{h} - 2 \overline{\omega'h'} \frac{\partial \overline{h}}{\partial z}.
\]

For large-scale averages, the time-rate-of-change term of (126) is negligible, as are the terms representing horizontal and vertical advection of \(\overline{h'^2}\) by the mean flow, as are the other terms involving horizontal derivatives, so that (126) can be drastically simplified to
\begin{equation}
0 \equiv -\frac{\partial (w'h')}{\partial z} - 2w'h' \frac{\partial \bar{h}}{\partial z}.
\end{equation}

(126)

c) Now suppose that the vertical velocity fluctuations represented by $w'$ are associated with a single family of cumulus updrafts covering fractional area. Show that

\begin{equation}
\bar{w'h'} = \sigma (1 - \sigma) (w_u - w_d) (h_u - h_d),
\end{equation}

(127)

\begin{equation}
\bar{w'h'h'} = \sigma (1 - \sigma) (1 - 2\sigma) (w_u - w_d) (h_u - h_d)^2,
\end{equation}

(128)

where $w_u$ and $w_d$ are the updraft and downdraft velocities, respectively, and $h_u$ and $h_d$ are the corresponding values of the moist static energy.

d) Define a “convective mass flux” by

\begin{equation}
M \equiv \sigma w_u.
\end{equation}

(129)

You may assume

\begin{equation}
\sigma \ll 1,
\end{equation}

(130)

and correspondingly that

\begin{equation}
w_u \gg w_d \text{ and } h_d \equiv \bar{h}.
\end{equation}

(131)

Show that if the convective mass flux is independent of height, then (127) can be approximated by

\begin{equation}
-\frac{\partial (w'h')}{\partial z} \equiv M \frac{\partial \bar{h}}{\partial z}.
\end{equation}

(132)

QuickStudies Referenced

Moist Adiabatic Lapse Rate

An Introduction to the General Circulation of the Atmosphere