CHAPTER 9  Vertical Differencing for Quasi-Static Models

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9.1 Introduction

Vertical differencing is a very different problem from horizontal differencing. This may seem odd, but the explanation is very simple. There are three primary factors. First, gravitational effects are very powerful, and act only in the vertical. Second, the Earth’s atmosphere is very shallow compared to its horizontal extent. Third, the atmosphere has a lower boundary.

To construct a vertically discrete model, we have to make a lot of choices, including these:

- The governing equations: Quasi-static or not? Shallow atmosphere or not? Anelastic or not?
- The vertical coordinate system
- The vertical staggering of the model’s dependent variables
- The properties of the exact equations that we want the discrete equations to mimic

As usual, these choices will involve trade-offs. Each possible choice will have strengths and weaknesses.

We must also be aware of possible interactions between the vertical differencing and the horizontal and temporal differencing.

9.2 Choice of equation set

The speed of sound in the Earth’s atmosphere is about 300 m s$^{-1}$. If we permit vertically propagating sound waves, then, with explicit time differencing, the largest time step that is compatible with linear computational stability can be quite small. For example, if a model has a vertical grid spacing on the order of 300 m, the allowed time step will be on the order of 1 second. This may be palatable if the horizontal and vertical grid spacings are comparable. On the other hand, with a horizontal grid spacing of 30 km and a vertical grid spacing of 300 m, vertically propagating sound waves will limit the time step to about one percent of the value that would be compatible with the horizontal grid spacing. That’s hard to take.
There are four possible ways around this problem. One approach is to use a set of equations that filters sound waves, i.e., “anelastic” equations. There are some issues with this approach, depending on the intended applications of the model, but anelastic models are very widely used and the anelastic equations can be an excellent choice for some applications, e.g., cloud modeling.

A second approach is to adopt the quasi-static system of equations, in which the equation of vertical motion is replaced by the hydrostatic equation. The quasi-static system of equations filters vertically propagating sound waves, while permitting Lamb waves, which are sound waves that propagate only in the horizontal. The quasi-static system is widely used in global models for both weather prediction and climate.

The third approach is to use implicit or partially implicit time differencing, which can permit a long time step even when vertically propagating sound waves occur. The main disadvantage is complexity.

The fourth approach is to “sub-cycle.” This means that small time steps are used to integrate the terms of the equations that govern sound waves, while longer time steps are used for the remaining terms.

### 9.3 General vertical coordinate

The most obvious choice of vertical coordinate system, and one of the least useful, is height. As you probably already know, the equations of motion are frequently expressed using vertical coordinates other than height. The most basic requirement for a variable to be used as a vertical coordinate is that it vary monotonically with height. Even this requirement can be relaxed; e.g., a vertical coordinate can be independent of height over some layer of the atmosphere, provided that the layer is not too deep.

Factors to be weighed in choosing a vertical coordinate system for a particular application include the following:

- the form of the lower boundary condition (simpler is better);
- the form of the continuity equation (simpler is better);
- the form of the horizontal pressure gradient force (simpler is better, and a pure gradient is particularly good);
- the form of the hydrostatic equation (simpler is better);
- the “vertical motion” seen in the coordinate system (less vertical motion is simpler and better);
- the method used to compute the vertical motion (simpler is better).

Each of these factors will be discussed below, for specific vertical coordinates. We begin, however, by presenting the basic governing equations, for quasi-static motions, using a general vertical coordinate.

Kasahara (1974) published a detailed discussion of general vertical coordinates for quasi-static models. A more modern discussion of the same subject is given by Konor and
Arakawa (1997). In a general vertical coordinate, $\zeta$, the quasi-static equation can be expressed as

$$\frac{\partial \phi}{\partial \zeta} = \left( \frac{\partial \phi}{\partial p} \right) \left( \frac{\partial p}{\partial \zeta} \right)$$

$$= \alpha m_\zeta,$$ (9.1)

where $\phi \equiv gz$ is the geopotential, $g$ is the acceleration of gravity, $z$ is height, $p$ is the pressure, and $\alpha$ is the specific volume. In deriving (9.1), we have used the hydrostatic equation in the form

$$\frac{\partial \phi}{\partial p} = -\alpha,$$ (9.2)

and we define

$$m_\zeta \equiv -\left( \frac{\partial p}{\partial \zeta} \right)$$ (9.3)

as the pseudo-density, i.e., the amount of mass (as measured by the pressure difference) between two $\zeta$-surfaces. The minus sign in (9.3) is arbitrary, and can be included or not according to taste, perhaps depending on the particular choice of $\zeta$.

The equation expressing conservation of an arbitrary intensive scalar, $\psi$, can be written as

$$\left( \frac{\partial}{\partial t} m_\zeta \psi \right)_\zeta + \nabla _\zeta \cdot \left( m_\zeta \mathbf{V} \psi \right) + \frac{\partial}{\partial p} (m_\zeta \dot{\zeta} \psi) = m_\zeta S_\psi.$$ (9.4)

Here

$$\dot{\zeta} \equiv \frac{D\zeta}{Dt}$$ (9.5)

is the rate of change of $\zeta$ following a particle, and $S_\psi$ is the source or sink of $\psi$, per unit mass. Eq. (9.4) can be derived by adding up the fluxes of $\psi$ across the boundaries of a control volume. We can obtain the continuity equation in $\zeta$-coordinates from (9.4), by putting $\psi \equiv 1$ and $S_\psi \equiv 0$:

$$\left( \frac{\partial m_\zeta}{\partial t} \right)_\zeta + \nabla _\zeta \cdot \left( m_\zeta \mathbf{V} \right) + \frac{\partial}{\partial p} (m_\zeta \dot{\zeta}) = 0.$$ (9.6)
By combining (9.4) and (9.6), we can obtain the advective form of the conservation equation for $\psi$:

$$m\zeta \frac{D\psi}{Dt} = S_\psi,$$

(9.7)

where the Lagrangian or material time derivative is expressed by

$$\frac{D}{Dt} = \left( \frac{\partial}{\partial t} \right)_\zeta + \mathbf{V} \cdot \nabla \zeta + \zeta \frac{\partial}{\partial \zeta}.$$

(9.8)

For example, the vertical pressure velocity,$$
\omega \equiv \frac{Dp}{Dt},
$$

(9.9)
can be written as

$$\omega = \left( \frac{\partial p}{\partial t} \right)_\zeta + \mathbf{V} \cdot \nabla \zeta p + \zeta \frac{\partial p}{\partial \zeta}$$

(9.10)

$$= \left( \frac{\partial p}{\partial t} \right)_\zeta + \mathbf{V} \cdot \nabla \zeta p - m\zeta \frac{\dot{\zeta}}{\zeta}.$$

The lower boundary condition, i.e., that no mass crosses the Earth’s surface, is expressed by requiring that a particle which is on the Earth’s surface remain there:

$$\frac{\partial \zeta}{\partial t} + \mathbf{V}_S \cdot \nabla \zeta_S - \dot{\zeta}_S = 0.$$

(9.11)

In the special case for which $\zeta_S$ is independent of time and the horizontal coordinates, (9.11) reduces to $\dot{\zeta}_S = 0$. Eq. (9.11) can actually be derived by integration of (9.6) throughout the entire atmospheric column, which gives

$$\frac{\partial}{\partial t} \int_{\zeta_S}^{\zeta_T} m\zeta \, ds + \nabla \cdot \left( \int_{\zeta_S}^{\zeta_T} m\zeta \mathbf{V} \, ds \right)$$

$$+ \left( \frac{\partial \zeta}{\partial t} \right)_S + \mathbf{V}_S \cdot \nabla \zeta_S - \dot{\zeta}_S \right) - \left( \frac{\partial \zeta}{\partial t} + \mathbf{V}_T \cdot \nabla \zeta_T - \dot{\zeta}_T \right) = 0.$$ 

(9.12)

Here $\zeta_T$ is the value of $\zeta$ at the top of the model atmosphere. We allow the possibility that the top of the model is placed at a finite height. Even if the top of the model is at the “top of the atmosphere,” i.e., at $p = 0$, the value of $\zeta_T$ may or may not be finite, depending on the
definition of $\zeta$. The quantity $\frac{\partial \zeta_S}{\partial t} + \mathbf{v} \cdot \nabla \zeta_S - \dot{\zeta}_S$, which is identical to the left-hand side of (9.11), represents the mass flux across the Earth’s surface. Similarly, $\frac{\partial \zeta_T}{\partial t} + \mathbf{V} \cdot \nabla \zeta_T - \dot{\zeta}_T$ represents the mass flux across the top of the atmosphere, which we assume to be zero, i.e.,

$$\frac{\partial \zeta_T}{\partial t} + \mathbf{V} \cdot \nabla \zeta_T - \dot{\zeta}_T = 0.$$  

(9.13)

If the top of the model is assumed to be a surface of constant $\zeta$, which is usually the case, then (9.13) reduces to $\dot{\zeta}_T = 0$.

Substituting (9.11) and (9.13) into (9.12), we find that

$$\frac{\partial}{\partial t} \int_{\zeta_s}^{\zeta_T} m_\zeta \, d\zeta + \nabla \cdot \left( \int_{\zeta_s}^{\zeta_T} m_\zeta \mathbf{v} \, d\zeta \right) = 0. $$  

(9.14)

In view of (9.3), this is equivalent to

$$\frac{\partial p_S}{\partial t} - \frac{\partial p_T}{\partial t} + \nabla \cdot \left( \int_{p_T}^{p_S} \mathbf{v} \, dp \right) = 0, $$  

(9.15)

which is the surface pressure tendency equation. Depending on the definitions of $\zeta$ and $\zeta_T$, it may or may not be appropriate to set $\frac{\partial p_T}{\partial t} = 0$. By analogy with the derivation of (9.15), we can show that the pressure tendency on an arbitrary $\zeta$-surface satisfies

$$\left( \frac{\partial p}{\partial t} \right)_{\zeta} - \frac{\partial p_T}{\partial t} + \nabla \cdot \left( \int_{\zeta}^{\zeta_T} m_\zeta \mathbf{v} \, d\zeta \right) - (m_\zeta \dot{\zeta})_\zeta = 0. $$  

(9.16)

The thermodynamic equation can be written as

$$c_p \left[ \left( \frac{\partial T}{\partial t} \right)_\zeta + \mathbf{v} \cdot \nabla \zeta_T + \dot{\zeta}_T \frac{\partial T}{\partial \zeta} \right] = \omega \alpha + Q,$$  

(9.17)

where $c_p$ is the specific heat of air at constant pressure, $\alpha$ is the specific volume, and $Q$ is the heating rate per unit mass. An alternative form of the thermodynamic equation is

$$\left( \frac{\partial \theta}{\partial t} \right)_\zeta + \mathbf{v} \cdot \nabla \zeta \theta + \dot{\zeta} \frac{\partial \theta}{\partial \zeta} = \frac{Q}{\Pi}, $$  

(9.18)
where

\[ \Pi \equiv c_p \frac{T}{\theta} = c_p \left( \frac{p}{p_0} \right)^\kappa, \] (9.19)

is the Exner function. In (9.19), \( \theta \) is the potential temperature; \( p_0 \) is a constant reference pressure, usually taken to be 1000 hPa, and \( \kappa = \frac{R}{c_p} \), where \( R \) is the gas constant.

### 9.3.1 The equation of motion and the HPGF

The horizontal momentum equation can be written as

\[ \frac{\partial V}{\partial t} + (V \cdot \nabla \zeta)V + \frac{\tau}{\zeta} = - \nabla_p \phi - f \times V + F. \] (9.20)

Here \(- \nabla_p \phi\) is the horizontal pressure-gradient force (HPGF), which is expressed as minus the gradient of the geopotential along an isobaric surface, and \( F \) is the friction vector. Using the relation

\[ \nabla_p = \nabla \zeta - (\nabla \zeta p) \frac{\partial}{\partial p} \]

\[ = \nabla \zeta + \frac{\nabla \zeta p}{m_\zeta} \frac{\partial}{\partial \zeta}, \] (9.21)

we can rewrite the HPGF as

\[ - \nabla_p \phi = - \nabla \zeta \phi - \frac{1}{m_\zeta} \frac{\partial \phi}{\partial \zeta}(\nabla \zeta p). \] (9.22)

In view of (9.1), this can be expressed as

\[ - \nabla_p \phi = - \nabla \zeta \phi - \alpha \nabla \zeta p. \] (9.23)

This is a nice result. For the special case \( \zeta \equiv z \), (9.24) reduces to \(- \nabla_p \phi = - \alpha \nabla \zeta p \), and for the special case \( \zeta = p \) it becomes \(- \nabla_p \phi = - \nabla \phi \). These are both very familiar.

Another useful form of the HPGF is expressed in terms of the Montgomery potential, which is defined by

\[ M \equiv c_p T + \phi. \] (9.24)

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For the special case in which $\zeta = \theta$, which will be discussed in detail later, the hydrostatic equation (9.1) can be written as

$$\frac{\partial M}{\partial \theta} = \Pi. \quad (9.25)$$

With the use of (9.24) and (9.25), Eq. (9.22) can be expressed as

$$-\nabla_p \phi = -\nabla_\zeta M + \Pi \nabla_\zeta \theta. \quad (9.26)$$

This form of the HPGF will be discussed later.

When the HPGF is a gradient, it has no effect in the vorticity equation, since the curl of the gradient is always zero. It is apparent from (9.23) and (9.26), however, that in general the HPGF is not simply a gradient. When the HPGF is not a gradient, it can spin up or spin down a circulation on a $\zeta$ surface. From (9.23) we see that the HPGF is a pure gradient for $\zeta = p$, and from (9.26) we see that the HPGF is a pure gradient for $\zeta = \theta$. This is an advantage shared by the pressure and theta coordinates.

The vertically integrated HPGF has a very important property that can be used in the design of vertical differencing schemes. With the use of (9.3), we can rewrite (9.22) as

$$-\nabla_p \phi = \frac{1}{m_\zeta} \nabla (m_\zeta \phi) - \frac{1}{m_\zeta \partial_\zeta} (\phi \nabla_\zeta p). \quad (9.27)$$

Vertically integrating with respect to mass, we find that

$$\int_{\zeta_T}^{\zeta_S} m_\zeta \nabla_p \phi d\zeta = -\nabla_\zeta \int_{\zeta_T}^{\zeta_S} m_\zeta \phi d\zeta - \phi_S \nabla p_S + \phi_T \nabla p_T. \quad (9.28)$$

Suppose that $\phi_T$ = constant or $\nabla p_T = 0$, and consider a line integral of the vertically integrated HPGF, i.e., $-\int_{\zeta_T}^{\zeta_S} m_\zeta \nabla_p \phi d\zeta$, along a closed path. It follows from (9.28) that the line integral must vanish if the surface geopotential is constant along the path of integration. In other words, in the absence of topography along the bounding path there cannot be any net spin-up or spin-down of a circulation in the region enclosed by the path. Later we will show how this important constraint can be mimicked in a vertically discrete model.

### 9.3.2 Vertical mass flux for a family of vertical coordinates

Konor and Arakawa (1997) derived a diagnostic equation that can be used to compute $\dot{\bar{\zeta}}$ for a large family of vertical coordinates that can be expressed as functions of the potential temperature, the pressure, and the surface pressure, i.e.,
While not completely general, Eq. (9.29) does cover a variety of interesting cases, which will be discussed below. By differentiating (9.29) with respect to time on a constant \( \zeta \) surface, we find that

\[
0 = \frac{\partial}{\partial t} F(\theta, p, p_S) \left[ \zeta \right].
\]  

(9.30)

The chain rule tells us that this is equivalent to

\[
\frac{\partial F}{\partial \theta} \left( \frac{\partial \theta}{\partial t} \right)_\zeta + \frac{\partial F}{\partial p} \left( \frac{\partial p}{\partial t} \right)_\zeta + \frac{\partial F}{\partial p_S} \frac{\partial p_S}{\partial t} = 0.
\]  

(9.31)

Substituting from (9.18), (9.16), and (9.15), we obtain

\[
\frac{\partial F}{\partial \theta} \left[ - \left( \mathbf{V} \cdot \nabla \zeta \theta + \zeta \frac{\partial \theta}{\partial \zeta} + \frac{Q}{\Pi} \right) \right]
\]
\[
+ \frac{\partial F}{\partial p} \left[ \frac{\partial p}{\partial t} - \nabla \cdot \left( \int_{\zeta}^{\zeta_T} m_\zeta \mathbf{V} \, d\zeta \right) + (m_\zeta \zeta) \right] \]
\[
+ \frac{\partial F}{\partial p_S} \left[ \frac{\partial p}{\partial t} - \nabla \cdot \left( \int_{p_T}^{p_S} m_\zeta \mathbf{V} \, dp \right) \right] = 0.
\]  

(9.32)

This can be solved for the vertical velocity, \( \zeta \):

\[
\left\{ \frac{\partial \theta \partial F}{\partial \zeta \partial \theta} - m_\zeta \frac{\partial F}{\partial \zeta} \right\} \zeta = \frac{\partial F}{\partial \theta} \left[ - \mathbf{V} \cdot \nabla \zeta \theta + \frac{Q}{\Pi} \right]
\]
\[
+ \frac{\partial F}{\partial p} \left[ \frac{\partial p}{\partial t} - \nabla \cdot \left( \int_{\zeta}^{\zeta_T} m_\zeta \mathbf{V} \, d\zeta \right) \right] \]
\[
+ \frac{\partial F}{\partial p_S} \left[ \frac{\partial p}{\partial t} - \nabla \cdot \left( \int_{p_T}^{p_S} m_\zeta \mathbf{V} \, dp \right) \right].
\]  

(9.33)

Here we have assumed that the heating rate, \( Q \), is not formulated as an explicit function of \( \zeta \).

As a check, consider the special case \( F \equiv p \), so that \( m = 1 \), and assume that
\[ \frac{\partial p_T}{\partial t} = 0, \]

as would be natural for the case of pressure coordinates. Then (9.33) reduces to

\[ p = \nabla \cdot \left( \int_{\zeta}^{z_f} m_p \mathbf{V} \, dp \right) \]

\[ = \nabla \cdot \left( \int_{p_f}^{p} \mathbf{V} \, dp \right). \] (9.34)

As a second special case, suppose that \( F \equiv \theta \). Then (9.33) becomes

\[ \dot{\theta} = \frac{Q}{\Pi}. \] (9.35)

Both of these are the expected results.

We assume that the model top is a surface of constant \( \zeta \), i.e., \( \zeta = \zeta_T \). Because (9.29) must apply at the model top, we can write

\[ \left( \frac{\partial F}{\partial \theta_T/\theta_T, p_T} \right) \frac{\partial \theta_T}{\partial t} + \left( \frac{\partial F}{\partial p_T/\theta_T, p_T} \right) \frac{\partial p_T}{\partial t} + \left( \frac{\partial F}{\partial p_S/\theta_T, p_T} \right) \frac{\partial p_S}{\partial t} = 0. \] (9.36)

Suppose that \( F(\theta, p, p_S) \) is chosen in such a way that \( \frac{\partial F}{\partial p_S/\theta_T, p_T} = 0 \), so that (9.36) simplifies to

\[ \left( \frac{\partial F}{\partial \theta_T/\theta_T, p_T} \right) \frac{\partial \theta_T}{\partial t} + \left( \frac{\partial F}{\partial p_T/\theta_T, p_T} \right) \frac{\partial p_T}{\partial t} = 0. \] (9.37)

Consider two possibilities. If we make the top of the model an isobaric surface, then \( \frac{\partial p_T}{\partial t} = 0 \), and also \( \frac{\partial F}{\partial \theta_T/\theta_T, p_T} = 0 \), so that (9.37) is trivially satisfied. Alternatively, if we make the top of the model an isentropic surface, then \( \frac{\partial \theta_T}{\partial t} = 0 \) and \( \left( \frac{\partial F}{\partial p_T/\theta_T, p_T} \right) \frac{\partial p_T}{\partial t} = 0 \), so again (9.37) is trivially satisfied.

Further discussion of how to choose the function \( F(\theta, p, p_S) \) is given later.

9.4 Discussion of particular vertical coordinate systems

We now discuss the following nine particular choices of \( \zeta \):

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• height, $z$

• pressure, $p$

• log-pressure, used in many theoretical studies

• sigma, defined by $\sigma \equiv \frac{p - p_T}{p_S - p_T}$, designed to simplify the lower boundary condition

• a “hybrid,” or “mix,” of sigma and pressure coordinates, used in numerous general circulation models including the forecast model of the European Centre for Medium Range Weather Forecasts

• eta, which is a modified sigma coordinate, defined by $\eta \equiv \frac{(p - p_T)}{(p_S - p_T)} \eta_S$, where $\eta_S$ is a time-independent function of the horizontal coordinates

• potential temperature, $\theta$, which has many attractive properties and is now being used

• entropy, $s = c_p \ln \theta$

• a hybrid sigma-theta coordinate, which behaves like sigma near the Earth’s surface, and like theta away from the Earth’s surface.

Of these nine possibilities, all except the height coordinate and the eta coordinate are members of the family of coordinates given by (9.29).

9.4.1 Height

In height coordinates, the hydrostatic equation is

$$\frac{\partial p}{\partial z} = -\rho g,$$  \hspace{1cm} (9.38)

where $\rho \equiv \frac{1}{\alpha}$ is the density. We can obtain (9.38) simply by flipping (9.2) over. For the case of the height coordinate, the pseudodensity reduces to $\rho g$, which is proportional to the ordinary or “true” density.

The continuity equation in height coordinates is
This equation is easy to understand, but it is mathematically complicated, in that it is nonlinear and involves the time derivative of a quantity that varies with height, namely the density:

The lower boundary condition in height coordinates is

$$\frac{\partial z_S}{\partial t} + \mathbf{V}_S \cdot \nabla z_S - w_S = 0.$$  \hfill (9.40)

Normally we can assume that $z_S$ is independent of time, but (9.40) can accommodate the effects of a specified time-dependent value of $z_S$ (e.g. to represent the effects of an earthquake, or a wave on the sea surface). Because height surfaces intersect the Earth’s surface, height-coordinates are relatively difficult to implement in numerical models. This complexity is mitigated somewhat by the fact that the horizontal spatial coordinates where the height surfaces meet the Earth’s surface are normally independent of time.

Note that (9.39) and (9.40) are direct transcriptions of (9.6) and (9.11), respectively, with the appropriate changes in notation.

The thermodynamic energy equation is

$$c_p \rho \left( \frac{\partial T}{\partial t} \right)_z = -c_p \rho \left( \mathbf{V} \cdot \nabla z T + w \frac{\partial T}{\partial z} \right) + \omega \alpha + Q.$$  \hfill (9.41)

Here $Q$ is the diabatic heating per unit volume, and

$$\omega = \left( \frac{\partial p}{\partial t} \right)_z + \mathbf{V} \cdot \nabla p + w \frac{\partial p}{\partial z}$$ \hfill (9.42)

$$= \left( \frac{\partial p}{\partial t} \right)_z + \mathbf{V} \cdot \nabla p - \rho g w .$$

By using (9.42) in (9.41), we find that

$$c_p \rho \left( \frac{\partial T}{\partial t} \right)_z = -c_p \rho \mathbf{V} \cdot \nabla z T - \rho w \Gamma_d + \Gamma T + \left( \frac{\partial p}{\partial t} \right)_z + \mathbf{V} \cdot \nabla p + Q ,$$ \hfill (9.43)

where the actual lapse rate and the dry-adiabatic lapse rate are given by

$$\Gamma = \frac{\partial T}{\partial z}.$$ \hfill (9.44)
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and

\[ \Gamma_d \equiv \frac{g}{c_p}, \]  

(9.45)

respectively. This form of the thermodynamic equation is awkward because it involves the
time derivatives of both \( T \) and \( p \). The time derivative of the pressure can be eliminated by
using the height-coordinate version of (9.16), which is

\[ \left[ \frac{\partial}{\partial t} p(z) \right]_z = -g \nabla_z \cdot \int_{-\infty}^{\infty} (\rho \mathbf{V}) \, dz + g \rho(z) w(z) + \frac{\partial p}{\partial t} . \]  

(9.46)

Substitution into (9.43) gives

\[ c_p \rho \left( \frac{\partial T}{\partial t} \right)_z = -c_p \rho \mathbf{V} \cdot \nabla_z T - \rho w c_p (\Gamma_d - \Gamma) \]

\[ + \left[ -g \nabla_z \cdot \int_{-\infty}^{\infty} (\rho \mathbf{V}) \, dz + g \rho(z) w(z) + \frac{\partial p}{\partial t} \right] + \mathbf{V} \cdot \nabla_z p + Q . \]  

(9.47)

According to (9.47), the time rate of change of the temperature at a given height is influenced
by the motion field through a deep layer. An alternative, considerably simpler form of the
thermodynamic energy equation is

\[ \left( \frac{\partial \theta}{\partial t} \right)_z = -\left( \mathbf{V} \cdot \nabla_z \theta + w \frac{\partial \theta}{\partial z} \right) + \frac{Q}{\Pi} . \]  

(9.48)

In quasi-static models using height coordinates, the equation of vertical motion is
replaced by the hydrostatic equation, in which \( w \) does not even appear. How then can we
compute \( w \)? The height coordinate is not a member of the family of schemes defined by
(9.29), and so (9.33), the formula for the vertical mass flux derived from (9.29), does not
apply. Instead, \( w \) is computed using “Richardson’s equation,” which is an expression of the
physical fact that hydrostatic balance applies not just at a particular instant, but continuously
through time. Richardson’s equation is actually closely analogous to (9.33), but somewhat
more complicated. The derivation of Richardson’s equation is also more complicated than the
derivation of (9.33).

The equation of state is

\[ p = \rho RT . \]  

(9.49)

Logarithmic differentiation of (9.49) gives

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The time derivatives can be eliminated by using (9.39), (9.46) and (9.47). After some manipulation, we find that

\[
\frac{1}{p} \frac{\partial p}{\partial t} = \frac{1}{\rho} \frac{\partial \rho}{\partial t} + \frac{1}{T} \frac{\partial T}{\partial t}. \tag{9.50}
\]

where

\[
c_p = cp - R
\]

is the specific heat of air at constant volume.

Eq. (9.51) can be simplified considerably as follows. Expand the vertical derivative term using the product rule:

\[
c_p T \frac{\partial}{\partial z} (\rho w) + \rho \left[ g \frac{c_v}{R} + c_p (\Gamma_d - \Gamma) \right] = \left(-c_p \rho V \cdot \nabla_z T + V \cdot \nabla_z p\right) - c_p T \nabla_z \cdot (\rho V) + g \frac{c_v}{R} \nabla_z \cdot \int (\rho V) \, dz + Q, \tag{9.51}
\]

where

\[
c_v \equiv c_p - R
\]

Logarithmic differentiation of the equation of state gives

\[
\frac{1}{p} \frac{\partial p}{\partial z} = \frac{1}{\rho} \frac{\partial \rho}{\partial z} + \frac{1}{T} \frac{\partial T}{\partial z}, \tag{9.54}
\]

which is equivalent to

\[
\frac{1}{p} \frac{\partial p}{\partial z} = -\frac{\rho g}{p} + \frac{\Gamma}{T} = \frac{1}{T} \left(-\frac{g}{R} + \Gamma\right). \tag{9.55}
\]

Substitute (9.55) into (9.53) to obtain

\[
\frac{c_p T \partial}{\rho \partial z} (\rho w) = c_p T \frac{\partial w}{\partial z} + w c_p \left(-\frac{g}{R} + \Gamma\right). \tag{9.56}
\]

Finally, substitute (9.56) into (9.51), and combine terms, to obtain
This is Richardson’s equation. It can be solved as a linear first-order ordinary differential equation for $w(z)$, given a lower boundary condition and the information needed to compute the various terms on the right-hand side, which involve both the mean horizontal motion and the heating rate. A physical interpretation of (9.57) is that the vertical motion is whatever it takes to maintain hydrostatic balance through time despite the fact that the various processes represented on the right-hand side of (9.57) may tend to upset that balance.

The complexity of Richardson’s equation has discouraged the use of height coordinates in quasi-static models; one of the very few exceptions was the early NCAR GCM (Kasahara and Washington, 1967).

As an example to illustrate the implications of (9.57), suppose that we have horizontally uniform heating but no horizontal motion. Then (9.57) drastically simplifies to

$$\rho c_p T \frac{\partial w}{\partial z} = Q.$$  \hspace{1cm} (9.58)

If the lower boundary is flat so that

$$w = 0 \text{ at } z = 0,$$  \hspace{1cm} (9.59)

then we obtain

$$w(z) = \int_0^z \frac{Q}{\rho c_p T} \, dz,$$  \hspace{1cm} (9.60)

i.e., heating (cooling) below a given level induces rising (sinking) motion at that level. The rising motion induced by heating below a given level can be interpreted as a manifestation of the upward movement of air particles as the air expands above the rigid lower boundary.

### 9.4.2 Pressure

The hydrostatic equation in pressure coordinates has already been stated; it is (9.2). The pseudo-density is simply unity, since (9.3) reduces to

$$m_p = 1.$$  \hspace{1cm} (9.61)

As a result, the continuity equation in pressure coordinates is relatively simple; it is linear and does not involve a time derivative:
\[ \nabla_p \cdot \mathbf{v} + \frac{\partial \omega}{\partial p} = 0. \] (9.62)

On the other hand, the lower boundary condition is complicated:

\[ \frac{\partial p_S}{\partial t} + \mathbf{v}_S \cdot \nabla p_S - \omega_S = 0. \] (9.63)

Recall that \( p_S \) can be predicted using the surface pressure-tendency equation, (9.15). Substitution from (9.15) into (9.63) gives

\[ \omega_S = \frac{\partial p_T}{\partial t} - \nabla \cdot \left( \int_{p_T}^{p_s} \mathbf{v} \, dp \right) + \mathbf{v}_S \cdot \nabla p_S, \] (9.64)

which can be used to diagnose \( \omega_S \). Nevertheless, the fact that pressure surfaces intersect the ground at locations that change with time (unlike height coordinates), means that models that uses pressure coordinates are complicated. Largely for this reason pressure coordinates are hardly ever used in numerical models.

With the pressure coordinate, we can write

\[ \frac{\partial}{\partial t} \left( \frac{\partial \phi}{\partial p} \right)_p = \frac{R}{p} \left( \frac{\partial T}{\partial t} \right)_p. \] (9.65)

This allows us to eliminate the temperature in favor of the geopotential, which is often done in theoretical studies.

### 9.4.3 Log-pressure

Obviously a surface of constant \( p \) is also a surface of constant \( \ln p \). Nevertheless, the equations take different forms in the \( p \) and \( \ln p \) coordinate systems.

Let \( T_0 \) be a constant reference temperature. Define the “log-pressure coordinate” \( z^* \) by the differential relationship

\[ dz^* = \frac{RT_0}{g} d(\ln p) = \frac{RT_0}{g} \frac{dp}{p}. \] (9.66)

Note that \( z^* \) has the units of length (i.e., height), and that \( dz^* = dz \) when \( T(p) = T_0 \). Although generally \( z \neq z^* \), we can force \( z(p = p_S) = z^*(p = p_S) \). From (9.66), we see that
where
\[ \phi^* \equiv gz^*. \] (9.68)

We also have the hydrostatic equation in the form
\[ \frac{\partial \phi}{\partial p} = \frac{RT}{p}. \] (9.69)

Subtracting (9.67) from (9.69), we obtain a useful form of the hydrostatic equation:
\[ \frac{\partial}{\partial p} (\phi - \phi^*) = -\frac{R(T - T_0)}{p}. \] (9.70)

Since \( \phi^* \) and \( T_0 \) are independent of time, we see that
\[ \frac{\partial}{\partial t} \left( \frac{\partial \phi}{\partial p} \right) = \frac{R}{p} \left( \frac{\partial T}{\partial t} \right)^*_{z^*}. \] (9.71)

### 9.4.4 The \( \sigma \) -coordinate

The \( \sigma \)-coordinate of Phillips (1957) is defined by
\[ \sigma \equiv \frac{p - p_T \pi}{\pi}, \] (9.72)

where we define
\[ \pi \equiv p_S - p_T, \] (9.73)

which is independent of height. Obviously,
\[ \sigma_S = 1 \text{ and } \sigma_T = 0. \] (9.74)

Note that for a fixed value of \( \sigma \)
\[ dp = \sigma d\pi, \] (9.75)

where the differential can represent a fluctuation in either time or (horizontal) space, with a fixed value of \( \sigma \). Also,
\[ \frac{\partial}{\partial p} (\cdot) = \frac{1}{\pi} \frac{\partial}{\partial \sigma} (\cdot). \] (9.76)
Here the differentials are evaluated at fixed horizontal position. The pseudodensity in \( \sigma \) -coordinates in \( \sigma \) -coordinates is

\[
m_{\sigma} = \pi, \tag{9.77}\]

which is independent of height. The continuity equation in \( \sigma \) -coordinates can therefore be written as

\[
\frac{\partial \pi}{\partial t} + \nabla_{\sigma} \cdot (\pi \mathbf{V}) + \frac{\partial}{\partial \sigma}(\pi \dot{\sigma}) = 0. \tag{9.78}\]

Although this equation does contain a time derivative, the differentiated quantity, \( \pi \), is independent of height, which makes (9.78) considerably simpler than (9.6).

The lower boundary condition in \( \sigma \) -coordinates is very simple:

\[
\dot{\sigma} = 0 \text{ at } \sigma = 1. \tag{9.79}\]

This simplicity was in fact Phillips’ motivation for the invention of \( \sigma \) -coordinates. The upper boundary condition is similar:

\[
\dot{\sigma} = 0 \text{ at } \sigma = 0. \tag{9.80}\]

The continuity equation in \( \sigma \) -coordinates plays a dual role. First, it is used to predict the surface pressure. This is done by integrating (9.78) through the depth of the vertical column and using the boundary conditions (9.79) and (9.80), to obtain the surface pressure-tendency equation in the form

\[
\frac{\partial \pi}{\partial t} = -\nabla \cdot \left( \int_{0}^{1} \pi \nabla_{\sigma} \mathbf{V} d\sigma \right). \tag{9.81}\]

The continuity equation is also used to determine \( \pi \dot{\sigma} \). Once \( \frac{\partial \pi}{\partial t} \) has been evaluated using (9.81), which does not involve \( \pi \dot{\sigma} \), we can substitute back into (9.78) to obtain

\[
\frac{\partial}{\partial \sigma}(\pi \dot{\sigma}) = \nabla \cdot \left( \int_{0}^{1} \pi \nabla_{\sigma} d\sigma \right) - \nabla_{\sigma} \cdot (\pi \mathbf{V}). \tag{9.82}\]

This can be integrated vertically to obtain \( \pi \dot{\sigma} \) as a function of \( \sigma \), starting from either the Earth’s surface or the top of the atmosphere, and using the appropriate boundary condition at the top or bottom. The same result is obtained regardless of the direction of integration.

The hydrostatic equation is simply
Finally, the horizontal pressure-gradient force takes a relatively complicated form:

\[
\text{HPGF} = -\sigma \alpha \nabla \pi - \nabla_{\sigma} \phi. \tag{9.84}
\]

Using the hydrostatic equation, (9.83), we can rewrite this as

\[
\text{HPGF} = \sigma \frac{1}{\pi} \frac{\partial \phi}{\partial \sigma} \nabla \pi - \nabla_{\sigma} \phi. \tag{9.85}
\]

Rearranging, we find that

\[
\pi(\text{HPGF}) = \sigma \frac{\partial \phi}{\partial \sigma} \nabla \pi - \nabla_{\sigma} \phi
\]

\[
= \left[ \frac{\partial}{\partial \sigma} (\sigma \phi) - \phi \right] \nabla \pi - \nabla_{\sigma} \phi. \tag{9.86}
\]

Vertically integrating (9.86) through the entire vertical column, we obtain

\[
\int_{0}^{1} \pi(\text{HPGF}) \, d\sigma = \phi_{S} \nabla \pi - \nabla_{\sigma} \left[ \int_{0}^{1} \pi \phi \, d\sigma \right]. \tag{9.87}
\]

When we integrate around any closed path, the second term on the right-hand side of (9.86) vanishes because it is the integral of a gradient. The first term also vanishes, unless there is topography along the path of integration. In short, the vertically integrated HPGF vanishes except in the presence of topography, in which case “mountain torque” may result. This conclusion is reached very easily when we start from (9.86).

Consider the two contributions to the HPGF when evaluated near a mountain, as illustrated in Fig. 9.1. Near steep topography, the spatial variations of \(p_{S}\) and the near-surface value of \(\phi\), along a \(\sigma\)-surface, are strong and of opposite sign. For example, moving uphill \(p_{S}\) decreases while \(\phi_{S}\) increases. As a result, the two terms on the right-hand side of (9.84) are individually large and opposing, and the HPGF is the relatively small difference between them -- a dangerous situation. In numerical models based on the \(\sigma\)-coordinate, near steep mountains the relatively small discretization errors in the individual terms of the right-hand side of (9.84) can be as large as the HPGF. This will be discussed further below.
9.5 More on the HPGF in \( \sigma \) -coordinates

Consider Fig. 9.2. At the point O, we have \( \sigma = \sigma^* \) and \( p = p^* \). We can write

\[
- \nabla_p \phi = - \nabla_\sigma \phi + (\nabla_\sigma \phi - \nabla_p \phi) = - \nabla_\sigma \phi + \nabla (\phi_{\sigma = \sigma^*} - \phi_{p = p^*}).
\] (9.88)

Compare with (9.84). Evidently

\[
- \sigma \alpha \nabla \pi = \nabla + (\phi_{\sigma = \sigma^*} - \phi_{p = p^*}).
\] (9.89)

The right-hand-side of (9.88) involves the gradient of the difference between \( \phi \) on a \( \sigma \) -surface and \( \phi \) on a \( p \) -surface. Computation of this difference in a vertically discrete model
Vertical Differencing for Quasi-Static Models

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amounts to vertical interpolation of $\phi$ from a $\sigma$-surface to a $p$-surface, and therefore should depend on the temperature through the hydrostatic equation. For a model that is discrete in both the horizontal and vertical, we must choose $\delta \sigma$ and $\delta x$ so that

$$\frac{\delta \sigma}{\delta x} \geq \frac{\langle \frac{\partial \phi}{\partial x} \rangle_\sigma}{\langle \frac{\partial \phi}{\partial \sigma} \rangle_x} \sim \text{rate of change of } \phi \text{ along a } \sigma\text{-surface} \quad \frac{\alpha}{p_s}. \quad (9.90)$$

The numerator of the right-hand side of (9.90) increases when the terrain is steep. The denominator increases when $T$ is warm, i.e. near the surface. The inequality (9.90) means that $\delta x$ must be fine enough for a given $\delta \sigma$; this shows that increasing the vertical resolution of a model $\sigma$-coordinate can cause problems unless the horizontal resolution is correspondingly increased. One way to minimize problems is to artificially smooth the topography.

9.5.1 Hybrid sigma-pressure coordinates

The advantage of the sigma coordinate is realized in the lower boundary condition. The disadvantage, in terms of the complicated and poorly behaved pressure-gradient force, is realized at all levels. This has motivated the use of hybrid coordinates that reduce to sigma at the lower boundary, and become pure pressure-coordinates at higher levels. In principle there are many ways of doing this. The most basic reference on this topic is the work of Simmons and Burridge (1981). They recommended the coordinate

$$\xi(p, p_S) \equiv \frac{P}{P_S} + \left( \frac{P}{P_S} - 1 \right) \frac{P - P_0}{P_S - P_0}, \quad (9.91)$$

where $P_0$ is specified as 1013.2 hPa. Inspection of (9.91) shows that $\xi = 1$ for $p = p_S$, and $\xi = 0$ for $p = 0$. Eq. (9.91) can be expanded and simplified to yield

$$\xi = \frac{P}{P_0} + \left( \frac{P}{P_S} \right)^2 \left( 1 - \frac{P_S}{P_0} \right). \quad (9.92)$$

Inspection of (9.92) shows that $\xi \rightarrow \frac{P}{P_0}$ as $\frac{P}{P_S} \rightarrow 0$. With (9.91), the pressure on an $\xi$-surface varies by less than one percent near the 10 mb level as the surface pressure varies in the range 1013 mb to 500 mb. When we evaluate the HPGF with the $\xi$-coordinate, there are still two terms, as with the $\sigma$-coordinate, but above the lower troposphere one of the terms is strongly dominant.

9.6 The $\eta$-coordinate

As a solution to the problem with the HPGF in $\sigma$-coordinates, Mesinger and Janjic (1985) proposed the $\eta$-coordinate, which is being used operationally at NCEP (the National
 Centers for Environmental Prediction):
\[ \eta = \sigma \eta_S \]  
(9.93)

where
\[ \eta_S = \frac{p_{rf}(z_S) - p_T}{p_{rf}(0) - p_T}. \]  
(9.94)

Whereas \( \sigma = 1 \) at the Earth's surface, Eq. (9.93) shows that \( \eta = \eta_S \) at the Earth's surface. According to (9.94), \( \eta_S = 1 \) (just as \( \sigma_S = 1 \)) if \( z_S = 0 \). Here \( z_S = 0 \) is chosen to be at or near “sea level.” The function \( p_{rf}(z_S) \) is pre-specified as a typical surface pressure for \( z = z_S \). Because \( z_S \) depends on the horizontal coordinates, \( p_{rf}(z_S) \) does too, and so, therefore, does \( \eta_S \). In fact, after choosing \( p_{rf}(z_S) \) and \( z_S(x,y) \), one can make a map of \( \eta_S(x,y) \), and of course this map is independent of time.

When we build a \( \sigma \)-coordinate model, we must specify (i.e., choose) fixed values of \( \sigma \) to serve as layer-edges and/or layer centers. Similarly, when we build an \( \eta \)-coordinate model, we must specify fixed values of \( \eta \) to serve as layer edges and/or layer centers. The values of \( \eta \) to be chosen include the possible values of \( \eta_S \). This means that only a few discrete choices of \( \eta_S \) are permitted; the number increases as the vertical resolution of the model increases. Mountains must come in a few discrete sizes, like off-the-rack clothing! This is sometimes called the “step-mountain” approach. Fig. 9.3 shows how the \( \eta \)-coordinate works near mountains. Note that, unlike \( \sigma \)-surfaces, \( \eta \)-surfaces are nearly flat in the sense that the pressure is nearly uniform on them. The circled u-points have \( u = 0 \), as a boundary condition on the sides of the mountains.

In \( \eta \)-coordinates, the HPGF still consists of two terms:
\[ -\nabla_p \phi = -\nabla_\eta \phi - \alpha \nabla_\eta p. \]  
(9.95)

Because the \( \eta \)-surfaces are nearly flat, however, these two terms are each comparable in magnitude to the HPGF itself, even near mountains, so the problem of near-cancellation does not occur.

### 9.6.1 Potential temperature

The potential temperature is defined by
\[ \theta = T \left( \frac{p_0}{p} \right)^\kappa. \]  
(9.96)
Potential temperature coordinates have particularly useful properties that have been recognized for many years, and have become more widely appreciated during the past decade or so. In the absence of heating, potential temperature is conserved following a particle. This means that the vertical motion in \( \theta \)-coordinates is proportional to the heating rate:

\[
\dot{\theta} = \frac{\theta}{c_p T} Q;
\]

in the absence of heating, there is “no vertical motion,” from the point of view of theta coordinates; we can also say that, in the absence of heating, a particle that is on a given theta surface remains on that surface. Eq. (9.97) is in fact an expression of the thermodynamic energy equation in \( \theta \)-coordinates. In fact, \( \theta \)-coordinates provide an especially simple pathway for the derivation of many important results, including the conservation equation for the Ertel potential vorticity. In addition, \( \theta \)-coordinates prove to have some important advantages for the design of numerical models (e.g. Eliassen and Raustein, 1968; Bleck, 1973; Johnson and Uccellini, 1983; Hsu and Arakawa, 1990).

The continuity equation in \( \theta \)-coordinates is given by

\[
\left( \frac{\partial m_\theta}{\partial t} \right)_\theta + \nabla_\theta \cdot (m_\theta \mathbf{V}) + \frac{\partial}{\partial p} (m_\theta \dot{\theta}) = 0,
\]
which is a direct transcription of (9.6). Note, however, that $\dot{\theta} = 0$ in the absence of heating; in such case, (9.98) reduces to

$$\left(\frac{\partial m_\theta}{\partial t}\right)_\theta + \nabla_\theta \cdot (m_\theta \mathbf{V}) = 0,$$  \hspace{1cm} (9.99)

which is analogous to the continuity equation of a shallow-water model.

The lower boundary condition in $\theta$-coordinates is

$$\frac{\partial \theta_S}{\partial t} + \mathbf{V}_S \cdot \nabla \theta_S - \dot{\theta}_S = 0.$$  \hspace{1cm} (9.100)

This equation must be used to predict $\theta_S$. The complexity of the lower boundary condition in $\theta$-coordinates is one of its chief drawbacks.

For the case of $\theta$-coordinates, the hydrostatic equation, (9.1), reduces to

$$\frac{\partial \phi}{\partial \theta} = -\alpha \frac{\partial \rho}{\partial \theta}.$$  \hspace{1cm} (9.101)

“Logarithmic differentiation” of (9.96) gives

$$\frac{d\theta}{\theta} = \frac{dT}{T} - \kappa \frac{dp}{p}.$$  \hspace{1cm} (9.102)

It follows that

$$\alpha \frac{\partial \rho}{\partial \theta} = c_p \frac{\partial T}{\theta} - \frac{T}{p}.$$  \hspace{1cm} (9.103)

Substitution of (9.103) into (9.101) gives

$$\frac{\partial M}{\partial \theta} = \Pi.$$  \hspace{1cm} (9.104)

The HPGF in $\theta$-coordinates is

$$\text{HPGF} = -\alpha \nabla_\theta \rho - \nabla_\theta \phi.$$  \hspace{1cm} (9.105)

From (9.96), we see that
It follows that

\[ \nabla \theta p = c_p \left( \frac{P}{RT} \right) \nabla T. \quad (9.107) \]

Substitution of (9.107) into (9.105) gives

\[ \text{HPGF} = -\nabla \theta M. \quad (9.108) \]

Of course, θ-surfaces can intersect the Earth’s surface, but we can consider these to follow the Earth’s surface, by defining imaginary “massless layers,” as shown in Fig. 9.4. Since no mass resides between the θ surfaces in the portion of the domain where they “touch the Earth’s surface,” no harm is done by this fantasy.

![Fig. 9.4: Four possible vertical coordinate systems.](image)

Of course, θ-surfaces can intersect the Earth’s surface, but we can consider these to follow the Earth’s surface, by defining imaginary “massless layers,” as shown in Fig. 9.4. Since no mass resides between the θ surfaces in the portion of the domain where they “touch the Earth’s surface,” no harm is done by this fantasy.

Obviously, a model that follows this approach has to be able to deal with massless layers. This practical difficulty has led most modelers to avoid θ-coordinates up to this time.

### 9.6.2 Entropy

The entropy coordinate is very similar to the θ-coordinate. We define the entropy by
9.6 The $\sigma$-coordinate

\[ s = c_p \ln \theta, \quad (9.109) \]

so that

\[ ds = c_p \frac{d\theta}{\theta}. \quad (9.110) \]

The hydrostatic equation can then be written as

\[ \frac{\partial M}{\partial s} = T. \quad (9.111) \]

This is a particularly attractive form because the “thickness” is simply given by the temperature.

9.6.3 Hybrid $\sigma - \theta$ coordinates

Konor and Arakawa (1997) discuss a hybrid vertical coordinate, $\zeta$, that reduces to $\theta$ away from the surface, and to $\sigma$ near the surface. This hybrid coordinate is a member of the family of schemes given by (9.29). It is designed to combine the strengths of $\theta$ and $\sigma$ coordinates, while avoiding their weaknesses. Hybrid coordinates have also been considered by other authors, e.g. Johnson and Uccellini (1983) and Zhu et al. (1992).

To specify the scheme, we must choose the function $F(\theta, p, p_S)$ that appears in (9.29). Following Konor and Arakawa (1997), define

\[ \zeta = F(\theta, p, p_S) \equiv f(\sigma) + g(\sigma)\theta, \quad (9.112) \]

where $\sigma \equiv \sigma(p, p_S)$ is a modified sigma coordinate, defined so that it is (as usual) a constant at the Earth’s surface, and (not as usual) increases upwards, e.g., $\sigma \equiv \frac{p_S - p}{p_S}$. If we specify $f(\sigma)$ and $g(\sigma)$, then the hybrid coordinate is fully determined.

We require, of course, that $\zeta$ itself increases upwards, so that

\[ \frac{\partial \zeta}{\partial \sigma} > 0. \quad (9.113) \]

We require that

\[ \zeta = \text{constant for } \sigma = \sigma_S, \quad (9.114) \]

which means that $\zeta$ is $\sigma$-like at the Earth’s surface, and also that

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which means that $\zeta$ becomes $\theta$ at the model top (or lower). These conditions imply, from (9.112), that

$$g(\sigma) \to 0 \text{ as } \sigma \to \sigma_S,$$

and $f(\sigma) \to 0$ and $g(\sigma) \to 0$ as $\sigma \to \sigma_T$. (9.117)

Now substitute (9.112) into (9.114), to obtain

$$\frac{\partial f}{\partial \sigma} + \frac{dg}{d\sigma} \theta + g \frac{\partial \theta}{\partial \sigma} > 0.$$ (9.118)

This is the requirement that $\zeta$ increases monotonically upward. Any choices for $f$ and $g$ that satisfy (9.116)-(9.118) can be used to define the hybrid coordinate.

Here is a way to do that: First, choose $g(\sigma)$ so that it is a monotonically increasing function of height, i.e.,

$$\frac{dg}{d\sigma} > 0 \text{ for all } \sigma.$$ (9.119)

We also choose $g(\sigma)$ so that the conditions (9.116)-(9.118) are satisfied. Obviously there are many possible choices for $g(\sigma)$ that will meet these requirements.

Next, define $\theta_{\text{min}}$ and $\left(\frac{\partial \theta}{\partial \sigma}\right)_{\text{min}}$ as lower bounds on $\theta$ and $\frac{\partial \theta}{\partial \sigma}$, respectively, i.e.,

$$\theta > \theta_{\text{min}} \text{ and } \frac{\partial \theta}{\partial \sigma} > \left(\frac{\partial \theta}{\partial \sigma}\right)_{\text{min}}.$$ (9.120)

When we choose the value of $\theta_{\text{min}}$, we are saying that we have no interest in simulating situations in which $\theta$ is actually colder than $\theta_{\text{min}}$. For example, we could choose $\theta_{\text{min}} = 10$ K. This is not necessarily an ideal choice, for reasons to be discussed below, but we can be sure that in our simulations $\theta$ will exceed 10 K everywhere at all times, unless the model is in the final throes of blowing up. Similarly, when we choose the value of $\left(\frac{\partial \theta}{\partial \sigma}\right)_{\text{min}}$, we are saying that we have no interest in simulating situations in which $\frac{\partial \theta}{\partial \sigma}$ is actually less stable.
(or more unstable) than \( \frac{\partial \theta}{\partial \sigma} \) \text{min}. We can choose \( \frac{\partial \theta}{\partial \sigma} < 0 \), i.e., a value of \( \frac{\partial \theta}{\partial \sigma} \) \text{min} that corresponds to a statically unstable sounding. Further discussion is given below.

Now, with reference to the inequality (9.118), we write the following equation:

\[
\frac{\partial f}{\partial \sigma} + \frac{dg}{d\sigma} \theta \text{min} + g \left( \frac{\partial \theta}{\partial \sigma} \right) \text{min} = 0. \tag{9.121}
\]

Recall that \( g(\sigma) \) has already been specified in such a way that (9.119) is satisfied. You should be able to see that if the equality (9.121) is satisfied, then the inequality (9.118) will also be satisfied, i.e., \( \zeta \) will increase monotonically upward. This will be true even if the sounding is statically unstable in some regions, provided that (9.120) is satisfied.

Eq. (9.121) is a first-order ordinary differential equation for \( f(\sigma) \), which must be solved subject to the boundary condition (9.117).

That’s all there is to it. Amazingly, the scheme does not involve any “if-tests.” It is simple and fairly flexible.

### 9.6.4 Summary of vertical coordinate systems

The table on the following page summarizes key properties of some important vertical coordinate systems. All of the systems discussed here (with the exception of the entropy coordinate) have been used in many theoretical and numerical studies. Each system has its advantages and disadvantages, which must be weighed with a particular application in mind.

At present, there seems to be a movement away from \( \sigma \) coordinates and towards \( \theta \) or hybrid \( \theta-\sigma \) coordinates.

The \( \theta \)-coordinate has many advantages. In the absence of heating, \( \frac{D\theta}{Dt} \equiv \dot{\theta} = 0 \), so there is “no vertical velocity.” This helps to minimize, for example, the problems associated with, e.g., the vertical advection of moisture. The HPGF has the simple form

\[
-\nabla_p \hat{\phi} = -\nabla_\theta (c_p T + gz),
\]

i.e. it is a gradient. The quantity

\[
M \equiv c_p T + \phi \tag{9.123}
\]

is called the “Montgomery potential” or “Montgomery stream function.” It satisfies a form of the hydrostatic equation that is natural for use with \( \theta \)-coordinates:

\[
\frac{\partial M}{\partial \theta} = c_p \left( \frac{P}{P_0} \right)^\kappa, \tag{9.124}
\]
<table>
<thead>
<tr>
<th>Coordinate</th>
<th>Hydrostatic</th>
<th>HPGF</th>
<th>Vertical Velocity</th>
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</thead>
<tbody>
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<td>$\frac{\partial p}{\partial z} = -\rho g$</td>
<td>$-\nabla \cdot \vec{p}$</td>
<td>$w = \frac{Dz}{Dt}$</td>
<td>$\frac{\partial p}{\partial t} + \nabla \cdot (\rho \vec{V}_H) + \frac{\partial}{\partial z} (\rho \vec{w}) = 0$</td>
<td>$V_S \cdot \nabla z_S - w_S = 0$</td>
</tr>
<tr>
<td>$P$</td>
<td>$\frac{\partial \phi}{\partial P} = -\alpha$</td>
<td>$-\nabla \cdot \vec{\phi}$</td>
<td>$w^* = \frac{Dz^*}{Dt}$</td>
<td>$\frac{\partial \phi}{\partial t} + \nabla \cdot (\sigma \vec{V}_H) + \frac{\partial}{\partial z^<em>} (\sigma \vec{w}^</em>) = 0$</td>
<td>$\frac{\partial \phi^<em>_S}{\partial t} + V_S \cdot \nabla \phi^</em>_S = 0$</td>
</tr>
<tr>
<td>$z^*$</td>
<td>$\frac{\partial z^*}{\partial z} = -\frac{T}{T_0}$</td>
<td>$-\nabla \cdot \vec{z}^*$</td>
<td>$\omega = \frac{\partial H}{\partial t}$</td>
<td>$\frac{\partial z^<em>_S}{\partial t} + V_S \cdot \nabla z^</em>_S = 0$</td>
<td>$-\frac{\partial \phi^<em>_S}{\partial t} + V_S \cdot \nabla \phi^</em>_S - \phi_S = 0$</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>$\frac{\partial z^*_S}{\partial z} = -\frac{T}{T_0}$</td>
<td>$-\nabla \cdot \vec{\sigma}$</td>
<td>$\frac{\partial \sigma}{\partial t} + \nabla \cdot (\sigma \vec{m}) = 0$</td>
<td>$\frac{\partial \sigma^<em>_S}{\partial t} + V_S \cdot \nabla \sigma^</em>_S - \sigma_S = 0$</td>
<td>$-\frac{\partial \phi^<em>_S}{\partial t} + V_S \cdot \nabla \phi^</em>_S - \phi_S = 0$</td>
</tr>
<tr>
<td>$q$</td>
<td>$\frac{\partial q}{\partial \Pi} = -\frac{T}{T_0}$</td>
<td>$-\nabla \cdot \vec{q}$</td>
<td>$\frac{\partial \phi}{\partial \Pi} = -\frac{T}{T_0}$</td>
<td>$\frac{\partial \phi^<em>_S}{\partial t} + V_S \cdot \nabla \phi^</em>_S = 0$</td>
<td>$-\frac{\partial \phi^<em>_S}{\partial t} + V_S \cdot \nabla \phi^</em>_S = 0$</td>
</tr>
<tr>
<td>$s$</td>
<td>$\frac{\partial \phi}{\partial s} = -\frac{T}{T_0}$</td>
<td>$-\nabla \cdot \vec{s}$</td>
<td>$\frac{\partial \phi}{\partial \mu} = -\frac{T}{T_0}$</td>
<td>$\frac{\partial \phi^*_S}{\partial t} = 0$</td>
<td>$-\frac{\partial \phi^<em>_S}{\partial t} + V_S \cdot \nabla \phi^</em>_S = 0$</td>
</tr>
</tbody>
</table>

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where $\kappa \equiv R/c_p$.

The dynamically important isentropic potential vorticity, $q$, is easily constructed in $\theta$-coordinates, since it involves the curl of $V$ on a $\theta$-surface:

$$q \equiv (\mathbf{k} \cdot \nabla_\theta \times \mathbf{V} + f) \frac{\partial \theta}{\partial p}.$$

(9.125)

The available potential energy is also easily obtained, since it involves the distribution of $p$ on $\theta$-surfaces.

### 9.7 Vertical staggering

After the choice of vertical coordinate system, the next issue is the choice of vertical staggering. Two possibilities are discussed here, and are illustrated in Fig. 9.5. These are the “Lorenz” or “L” grid, and the “Charney-Phillips” or “C-P” grid. Suppose that both grids have $N$ wind-levels. The L-grid also has $N$ $\theta$-levels, while the C-P grid has $N+1$ $\theta$-levels. On both grids, $\phi$ is hydrostatically determined on the wind-levels, and

$$\phi_i - \phi_{i+1} \sim \theta_{i+\frac{1}{2}}.$$

(9.126)

(Exercise: Show that $\partial \phi \Delta \Pi = -\theta$, where $\Pi \equiv c_p (p/p_0)^\kappa$.)

On the C-P grid, $\theta$ is located between $\phi$-levels, so (9.126) is convenient. With the L-grid, $\theta$ must be interpolated, e.g.

$$\phi_i - \phi_{i+1} \sim \frac{1}{2} (\theta_i + \theta_{i+1}).$$

(9.127)

Because (9.127) involves averaging, an oscillation in $\theta$ is not “felt” by $\phi$, and so has no effect on the winds. This allows the possibility of a computational mode in the vertical. No such problem occurs with the C-P grid.

There is a second, less obvious problem with the L grid. The vertically discrete potential vorticity corresponding to (9.125) is

$$q_l \equiv (\mathbf{k} \cdot \nabla_\theta \times \mathbf{V}_l + f) \left( \frac{\partial \theta}{\partial p} \right)_l.$$

(9.128)

It is obvious that (9.128) “wants” the potential temperature to be defined at levels “in between” the wind levels, as they are on the C-P grid. In contrast, on the L grid the potential temperature and wind are defined at the same level. Suppose that we have $N$ wind levels. Then with the C-P grid we will have $N+1$ potential temperature levels and $N$ potential vorticities. This is nice. With the L grid, on the other hand, it can be shown that we effectively have $N+1$ potential vorticities. The “extra” degree of freedom in the potential vorticity is
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spurious, and allows a kind of computational baroclinic instability (Arakawa and Moorthi, 1988). This is a drawback of the L grid.

As Lorenz (1955) pointed out, however, the L-grid is very convenient for maintaining total energy conservation, because the kinetic and thermodynamic energies are defined at the same levels. Today, almost all models use the L-grid. This may change.

9.8 Conservation properties of vertically discrete models using \( \sigma \)-coordinates

We now investigate conservation properties of the vertically discretized equations, using \( \sigma \)-coordinates, and using the L-grid. The discussion follows Arakawa and Lamb (1977), although some of the ideas originated with Lorenz (1960). For simplicity, we consider only vertical discretization, and keep the temporal and horizontal derivatives in continuous form.

Conservation of mass is expressed, in the vertically discrete system, by

\[
\frac{\partial \pi}{\partial t} + \nabla \mathbf{\cdot} \left( \pi \mathbf{V} \right) + \left[ \frac{\delta(\pi \dot{\sigma})}{\delta \sigma} \right]_l = 0, \tag{9.129}
\]

where

\[
\left[ \delta(\ ) \right]_l \equiv \left( \right)_{l+\frac{1}{2}} - \left( \right)_{l-\frac{1}{2}}. \tag{9.130}
\]
9.8 Conservation properties of vertically discrete models using \( \sigma \)-coordinates

Summing (9.129) over all levels, and using the boundary conditions

\[
\sigma_{\frac{1}{2}} = \sigma_{L+\frac{1}{2}} = 0, \tag{9.131}
\]

and

\[
\sum_{l=1}^{L} \delta \sigma_l = 1, \tag{9.132}
\]

we obtain

\[
\frac{\partial \pi}{\partial t} + \nabla \cdot \left( \sum_{l=1}^{L} (\pi \nabla_l)(\delta \sigma_l) \right) = 0, \tag{9.133}
\]

which is the vertically discrete form of the surface pressure tendency equation. From (9.133), we see that mass is, in fact, conserved, i.e., the vertical mass fluxes do not produce any net source or sink of mass.

We use

\[
p_{l+\frac{1}{2}} = \pi \sigma_{l+\frac{1}{2}} + p_T, \tag{9.134}
\]

where \( p_T \) is a constant, and the constant values of \( \sigma_{l+\frac{1}{2}} \) are prescribed for each layer edge when the model is started up. Eq. (9.134) tells how to compute layer-edge pressures. The method to discuss layer-center pressures will be discussed later.

Similarly, we can conserve an intensive scalar, such as the potential temperature \( \theta \), by using

\[
\frac{\partial}{\partial t} (\pi \theta_l) + \nabla \cdot (\pi \nabla_l \theta_l) + \left[ \frac{\delta (\pi \theta)}{\delta \sigma} \right]_{\theta} = 0. \tag{9.135}
\]

In order to use (9.135) it is necessary to define values of \( \theta \) at the layer edges, via an interpolation. We have already discussed the interpolation issue in the context of advection, and that discussion applies to vertical advection as well as horizontal advection. In particular, the interpolation methods that allow conservation of an arbitrary function of the advected quantity can be used for vertical advection.

Now refer back to the discussion of the horizontal pressure-gradient force, in connection with (9.86) and (9.87). A finite-difference analog of (9.86) is
Multiplying (9.136) by \( \delta \sigma \), and summing over all layers, we obtain

\[
\sum_{l=1}^{L} \pi(\text{HPGF})_l(\delta \sigma)_l = \left[ \frac{\delta (\sigma \phi)}{\delta \sigma} \right]_l \nabla \pi - \nabla (\pi \phi)_l.
\]

(9.137)

which is a finite-difference analog of (9.87). This means that if we use the form of the HPGF given by (9.136), the vertically summed HPGF cannot generate a circulation inside a closed path, in the absence of topography (Arakawa and Lamb, 1977). This “principle” provides a rational way to choose which of the many possible forms of the HPGF should be used in the model. At this point, of course, the form is not fully determined, because we do not yet have a method to compute either \( \phi_l \) or the layer-edge values of \( \phi \) that appear in (9.136). A suitable method is derived below.

Eq. (9.136) is equivalent to

\[
\pi(\text{HPGF})_l = \left[ \frac{\delta (\sigma \phi)}{\delta \sigma} \right]_l - \phi_l \nabla \pi - \pi \nabla \phi_l.
\]

(9.138)

By comparison with (9.84), we identify

\[
p_S(\sigma \alpha)_l = \phi_l - \left[ \frac{\delta (\sigma \phi)}{\delta \sigma} \right]_l.
\]

(9.139)

This will be used later.

Next consider total energy conservation. We begin by reviewing the continuous case. Potential temperature conservation is expressed by

\[
\frac{\partial}{\partial t} (\pi \theta) + \nabla \cdot (\pi V \theta) + \frac{\partial}{\partial \sigma} (\pi \sigma \theta) = 0
\]

(9.140)

Here we assume no heating for simplicity. Using continuity this can be expressed in advective form:

\[
\frac{\partial \theta}{\partial t} + V \cdot \nabla \theta + \sigma \frac{\partial \theta}{\partial \sigma} = 0.
\]

(9.141)
With the use of the definition of $\theta$, i.e.,

$$\theta = T \left( \frac{P_0}{P} \right)^{\kappa},$$

(9.142)

and the equation of state, (9.141) can be used to derive the thermodynamic energy equation in the form

$$c_p \left( \frac{\partial T}{\partial t} + \mathbf{V} \cdot \nabla T + \sigma \frac{\partial T}{\partial \sigma} \right) = \sigma \omega \alpha.$$  

(9.143)

Here

$$\omega \equiv \left( \frac{\partial p}{\partial t} \right)_\sigma + \mathbf{V} \cdot \nabla \sigma \rho + \sigma \frac{\partial p}{\partial \sigma}$$

$$= \sigma \left( \frac{\partial \pi}{\partial t} + \mathbf{V} \cdot \nabla \rho \sigma \right) + \pi \dot{\sigma}.$$  

(9.144)

Continuity then allows us to transform (9.143) to the flux form:

$$\frac{\partial}{\partial t} (\pi c_p T) + \nabla \cdot (\pi \mathbf{V} c_p T) + \frac{\partial}{\partial \sigma} (\pi \sigma c_p T) = \pi \omega \alpha.$$ 

(9.145)

The potential temperature equation, (9.140), is approximated by (9.135), which has already been discussed. Suppose that the model explicitly predicts $\theta_l$. Adopting the definition

$$\theta_l = \frac{T_l}{P_l},$$  

(9.146)

where for convenience we define

$$P_l \equiv \left( \frac{P_l}{P_0} \right)^{\kappa},$$ 

(9.147)

we will now derive a finite-difference analog of (9.140), by starting from (9.135). Recall that the method to determine $p_l$ has not been specified yet. The advective form corresponding to (9.135) is

$$\pi \left( \frac{\partial \theta_l}{\partial t} + \mathbf{V}_l \cdot \nabla \theta_l \right) + \frac{1}{(\delta \sigma)_l} \left[ (\pi \sigma)_l \left( \theta_{l+1} - \theta_{l-1} \right) + (\pi \sigma)_l \left( \theta_{l+1} - \theta_{l-1} \right) \right] = 0.$$
Substitute (9.146) into (9.148), to obtain the corresponding prediction equation for $T_i$:

$$\pi \left( \frac{\partial T_i}{\partial t} + \mathbf{V}_i \cdot \nabla T_i \right) - \frac{T_i \partial P_i}{P_i \partial \pi} \left( \frac{\partial \pi}{\partial t} + \mathbf{V}_i \cdot \nabla \pi \right)$$

$$+ \frac{1}{(T_i \delta \sigma)_l} \left[ (\pi \delta \sigma)_l \left( T_i + \frac{1}{2} \left( P_i \theta - T_i \right) \right) + (\pi \delta \sigma)_l \left( T_i - \frac{1}{2} \left( P_i \theta - T_i \right) \right) \right] = 0.$$  \hspace{1cm} (9.149)

The derivative $\frac{\partial P_i}{\partial \pi}$ cannot be evaluated until we specify the form of $P_i$. We now introduce the layer-edge temperatures, i.e., $T_{l+\frac{1}{2}}$ and $T_{l-\frac{1}{2}}$, although the method to determine them has not yet been specified. We rewrite (9.149) as

$$\pi \left( \frac{\partial T_i}{\partial t} + \mathbf{V}_i \cdot \nabla T_i \right) + \frac{1}{(T_i \delta \sigma)_l} \left[ (\pi \delta \sigma)_l \left( T_i + \frac{1}{2} \left( P_i \theta - T_i \right) \right) + (\pi \delta \sigma)_l \left( T_i - \frac{1}{2} \left( P_i \theta - T_i \right) \right) \right]$$

$$= \frac{T_i \partial P_i}{P_i \partial \pi} \left( \frac{\partial \pi}{\partial t} + \mathbf{V}_i \cdot \nabla \pi \right)$$

$$+ \frac{1}{(T_i \delta \sigma)_l} \left[ (\pi \delta \sigma)_l \left( T_i + \frac{1}{2} \left( P_i \theta - T_i \right) \right) + (\pi \delta \sigma)_l \left( T_i - \frac{1}{2} \left( P_i \theta - T_i \right) \right) \right].$$ \hspace{1cm} (9.150)

Obviously the left-hand side of (9.150) can be rewritten in flux form through the use of the vertically discrete continuity equation:

$$\frac{\partial}{\partial t} (\pi T_i) + \mathbf{V} \cdot (\pi \mathbf{V}_i T_i) + (\delta(\pi \delta \sigma) T_i) = \frac{T_i \partial P_i}{P_i \partial \pi} \left( \frac{\partial \pi}{\partial t} + \mathbf{V}_i \cdot \nabla \pi \right)$$

$$+ \frac{1}{(T_i \delta \sigma)_l} \left[ (\pi \delta \sigma)_l \left( T_i + \frac{1}{2} \left( P_i \theta - T_i \right) \right) + (\pi \delta \sigma)_l \left( T_i - \frac{1}{2} \left( P_i \theta - T_i \right) \right) \right].$$ \hspace{1cm} (9.151)

By comparison of (9.145) with (9.151), we identify
2159.8 Conservation properties of vertically discrete models using \( - \)coordinates

This result will be used below.

Returning to the continuous case, we now derive the continuous mechanical energy equation, starting from the continuous momentum equation in the form

\[
\frac{\partial \mathbf{V}}{\partial t} + [f + \mathbf{k} \cdot (\nabla \sigma \times \mathbf{V})] \mathbf{k} \times \mathbf{V} + \sigma \frac{\partial \mathbf{V}}{\partial \sigma} + \nabla \sigma K = - \nabla \sigma \phi - \sigma \alpha \nabla \pi .
\] (9.153)

Here \( K \equiv \frac{1}{2} \mathbf{V} \cdot \mathbf{V} \) is the kinetic energy per unit mass. Dotting (9.153) with \( \mathbf{V} \) gives the mechanical energy equation in the form

\[
\left( \frac{\partial K}{\partial t} \right)_{\sigma} + \nabla \sigma K = - \mathbf{V} \cdot (\nabla \sigma \phi + \sigma \alpha \nabla \pi) .
\] (9.154)

The corresponding flux form is

\[
\frac{\partial}{\partial t} (\pi K) + \nabla \cdot (\pi \mathbf{V} K) + \frac{\partial}{\partial \sigma} (\pi \sigma K) = - \pi \mathbf{V} \cdot (\nabla \sigma \phi + \sigma \alpha \nabla \pi) .
\] (9.155)

The pressure-work term on the right-hand side of (9.155) has to be manipulated to facilitate comparison with (9.145). Begin as follows:

\[
-\pi \mathbf{V} \cdot (\nabla \sigma \phi + \sigma \alpha \nabla \pi) = - \nabla \sigma \cdot (\pi \mathbf{V} \phi) + \phi \nabla \sigma \cdot (\pi \mathbf{V}) - \pi \sigma \alpha \mathbf{V} \cdot \nabla \pi
\]

\[
= - \nabla \sigma \cdot (\pi \mathbf{V} \phi) - \phi \left[ \frac{\partial \pi}{\partial t} + \frac{\partial}{\partial \sigma} (\pi \sigma) \right] - \pi \sigma \alpha \mathbf{V} \cdot \nabla \pi
\]

\[
= - \nabla \sigma \cdot (\pi \mathbf{V} \phi) - \frac{\partial}{\partial \sigma} (\pi \sigma \phi) + \pi \sigma \frac{\partial \phi}{\partial t} - \frac{\partial \pi}{\partial t} - \pi \sigma \alpha \mathbf{V} \cdot \nabla \pi
\] (9.156)

In the final line of (9.156) we have used hydrostatics. Referring back to (9.144), we can write

\[
\pi \sigma \alpha \pi + \phi \frac{\partial \pi}{\partial t} + \pi \sigma \alpha \mathbf{V} \cdot \nabla \pi = \pi \omega \alpha + \frac{\partial}{\partial \sigma} (\phi \sigma \frac{\partial \pi}{\partial t}) .
\] (9.157)
Substitution of (9.157) into (9.156) gives

\[ -\pi \mathbf{V} \cdot (\nabla \sigma \phi + \sigma \alpha \nabla \pi) = -\nabla \sigma \cdot (\pi \mathbf{V} \phi) - \frac{\partial}{\partial \sigma} \left( \pi \sigma \phi + \phi \sigma \frac{\partial \pi}{\partial t} \right) - \pi \omega \alpha. \tag{9.158} \]

Finally, plugging (9.158) back into (9.155), and collecting terms, gives the mechanical energy equation in the form

\[ \frac{\partial}{\partial t} (\pi K) + \nabla \cdot [\pi \mathbf{V} (K + \phi)] + \frac{\partial}{\partial \sigma} \left[ \pi \sigma (K + \phi) + \phi \sigma \frac{\partial \pi}{\partial t} \right] = -\pi \omega \alpha. \tag{9.159} \]

Adding (9.145) and (9.159) gives a statement of the conservation of total energy:

\[ \frac{\partial}{\partial t} [\pi (K + c_p T)] + \nabla \cdot [\pi \mathbf{V} (K + \phi + c_p T)] + \frac{\partial}{\partial \sigma} \left[ \pi \sigma (K + \phi + c_p T) + \phi \sigma \frac{\partial \pi}{\partial t} \right] = 0. \tag{9.160} \]

Integrating this through the depth of an atmospheric column, we find that

\[ \frac{\partial}{\partial t} \left[ \int_0^1 \pi (K + c_p T) d\sigma \right] + \nabla \cdot \left[ \int_0^1 \pi \mathbf{V} (K + \phi + c_p T) d\sigma \right] + \phi_s \frac{\partial \pi}{\partial t} = 0, \tag{9.161} \]

which can also be written as

\[ \frac{\partial}{\partial t} \left[ \int_0^1 \pi (K + c_p T + \phi_s) d\sigma \right] + \nabla \cdot \left[ \int_0^1 \pi \mathbf{V} (K + \phi + c_p T) d\sigma \right] = \pi \frac{\partial \phi_s}{\partial t}. \tag{9.162} \]

The right-hand side of (9.162) represents the work done on the atmosphere if the lower boundary is moving with time, e.g., in an earthquake.

We now carry out essentially the same derivation using the vertically discrete system. Taking the dot product of \( \pi \mathbf{V}_l \) with the HPGF for layer \( l \), we write, closely following (9.156)-(9.158),
\[-\pi \mathbf{V}_i \cdot [\nabla \phi_i + (\sigma \alpha)_i \nabla \pi] = -\nabla \cdot (\pi \mathbf{V}_i \phi_i) + \phi_i \nabla \cdot (\pi \mathbf{V}_i) - \pi (\sigma \alpha)_i \mathbf{V}_i \cdot \nabla \pi \]
\[= -\nabla \cdot (\pi \mathbf{V}_i \phi_i) - \phi_i \left( \frac{\partial \pi}{\partial t} + \frac{\delta (\pi \sigma)}{\delta \sigma} \right) - \pi (\sigma \alpha)_i \mathbf{V}_i \cdot \nabla \pi \]
\[= -\nabla \cdot (\pi \mathbf{V}_i \phi_i) \left. - \frac{\delta (\pi \sigma \phi)}{\delta \sigma} \right|_l \]
\[+ \frac{1}{(\delta \sigma)_l} \left( (\pi \sigma)_l + \frac{1}{2} \left( \phi_{l+1/2} - \phi_{l-1/2} \right) + (\pi \sigma)_l \right) \left( \phi_{l-1/2} - \phi_{l+1/2} \right) \right] \]
\[-\phi_i \frac{\partial \pi}{\partial t} - \pi (\sigma \alpha)_i \mathbf{V}_i \cdot \nabla \pi . \]

Continuing down this path, we construct the terms that we need by adding and subtracting
\[-\pi \mathbf{V}_i \cdot [\nabla \phi_i + (\sigma \alpha)_i \nabla \pi] = -\nabla \cdot (\pi \mathbf{V}_i \phi_i) - \left[ \delta (\pi \sigma \phi) \right] \]
\[+ \left[ (\pi \sigma)_l - \phi_i \right] \frac{\partial \pi}{\partial t} - \pi \left[ (\sigma \alpha)_l \left( \frac{\partial \pi}{\partial t} + \mathbf{V}_i \cdot \nabla \pi \right) \right] \]
\[\left[ \pi (\delta \sigma)_l \left( (\pi \sigma)_l + \frac{1}{2} \left( \phi_{l+1/2} - \phi_{l-1/2} \right) + (\pi \sigma)_l \right) \left( \phi_{l-1/2} - \phi_{l+1/2} \right) \right] . \]

Using (9.139) in the form
\[\pi (\sigma \alpha)_l - \phi_i = \left[ \delta (\sigma \phi) \right] \]
we can rewrite this as
\[-\pi \mathbf{V}_i \cdot [\nabla \phi_i + (\sigma \alpha)_i \nabla \pi] = -\nabla \cdot (\pi \mathbf{V}_i \phi_i) - \left[ \frac{\delta \left( \pi \sigma + \frac{\partial \pi}{\partial t} \phi \right)}{\delta \sigma} \right] \]
\[-\pi \left[ (\sigma \alpha)_l \left( \frac{\partial \pi}{\partial t} + \mathbf{V}_i \cdot \nabla \pi \right) \right] - \left[ \frac{(\pi \sigma)_l + \frac{1}{2} \left( \phi_{l+1/2} - \phi_{l-1/2} \right) + (\pi \sigma)_l \left( \phi_{l-1/2} - \phi_{l+1/2} \right)}{\pi (\delta \sigma)_l} \right] . \]

By comparing with (9.158), we infer that
\[
\pi(\omega \alpha)_l = \pi(\sigma \alpha)_l \left( \frac{\partial \pi}{\partial t} + \mathbf{V}_l \cdot \nabla \pi \right)
\]

\[
- \left[ (\pi \sigma)_{l+\frac{1}{2}} \left( \phi_{l+\frac{1}{2}} - \phi_l \right) + (\pi \sigma)_{l-\frac{1}{2}} \left( \phi_l - \phi_{l-\frac{1}{2}} \right) \right] (\delta \sigma)_l .
\]

(9.167)

We have now reached the crux of the problem. In order to ensure total energy conservation, the form of \( p \delta(\omega \alpha)_l \) given by (9.167) must match that given by (9.152). In order for this to happen, we need the following conditions to be satisfied:

\[
(\sigma \alpha)_l = \frac{T_i \partial P_l}{c_p \partial \pi} ,
\]

(9.168)

\[
\phi_l - \phi_{l+\frac{1}{2}} = c_p \left( T_{l+\frac{1}{2}} - P_l \theta_{l+\frac{1}{2}} \right) ,
\]

(9.169)

\[
\phi_{l-\frac{1}{2}} - \phi_l = c_p \left( P_l \theta_{l-\frac{1}{2}} - T_{l-\frac{1}{2}} \right) .
\]

(9.170)

Eq. (9.168) gives an expression for \((\sigma \alpha)_l\). We already had one, though, in Eq. (9.139). Requiring that these two formulae agree, we obtain

\[
\phi_l - \left[ \frac{\delta(\sigma \phi)}{\delta \sigma} \right]_l = c_p \frac{T_i \partial P_l}{P_l \partial \pi} .
\]

(9.171)

This is a finite-difference form of the hydrostatic equation.

With the use of Eq. (9.146), Eqs. (9.169)-(9.170) can be rewritten as

\[
\left( c_p T_{l+\frac{1}{2}} + \phi_{l+\frac{1}{2}} \right) - (c_p T_l + \phi_l) = P_l c_p \left( \theta_{l+\frac{1}{2}} - \theta_l \right) ,
\]

(9.172)

and

\[
(c_p T_{l-\frac{1}{2}} + \phi_{l-\frac{1}{2}}) - \left( c_p T_{l-\frac{1}{2}} + \phi_{l-\frac{1}{2}} \right) = P_l c_p \left( \theta_l - \theta_{l-\frac{1}{2}} \right) .
\]

(9.173)
respectively. These are also finite-difference analogs of the hydrostatic equation. The subscripts in these equations are arbitrary. Add one to each subscript in (9.173), and add the result to (9.172). This yields

\[ \phi_l - \phi_{l+1} = c_p \left( P_{l+1} - P_l \right) \theta_{l+\frac{1}{2}}. \]  

(9.174)

If the forms of \( P_l \) and \( \theta_{l+\frac{1}{2}} \) are specified, we can use (9.174) to integrate the hydrostatic equation upward from level \( l+1 \) to level \( l \).

It is still necessary, however, to determine the value of \( \phi_L \), i.e., the layer-center geopotential for the lowest model layer. This can be done by first summing \((\delta \sigma)_l \) times (9.171) over all layers:

\[ \sum_{l=1}^{L} \phi_l (\delta \sigma)_l - \phi_S = \sum_{l=1}^{L} \pi c_p T_l \frac{\partial P_l}{\partial \pi} (\delta \sigma)_l. \]  

(9.175)

But

\[ \sum_{l=1}^{L} \phi_l (\delta \sigma)_l = \sum_{l=1}^{L} \phi_l \left( \sigma_{l+\frac{1}{2}} - \sigma_{l-\frac{1}{2}} \right) \]

\[ = \phi_L + \sum_{l=1}^{L-1} \sigma_{l+\frac{1}{2}} (\phi_l - \phi_{l+1}) \]  

(9.176)

\[ = \phi_L + \sum_{l=1}^{L-1} \sigma_{l+\frac{1}{2}} c_p (P_{l+1} - P_l) \theta_{l+\frac{1}{2}}. \]

This is an identity. We can therefore write

\[ \phi_L = \phi_S + \sum_{l=1}^{L} \pi c_p T_l \frac{\partial P_l}{\partial \pi} (\delta \sigma)_l - \sum_{l=1}^{L-1} \sigma_{l+\frac{1}{2}} c_p (P_{l+1} - P_l) \theta_{l+\frac{1}{2}}. \]  

(9.177)

This is a bit odd, because it says that the thickness between the Earth’s surface and the middle of the lowest model layer depends on \( \theta \) at all levels in the entire column. From a mathematical point of view there is nothing wrong with that. In effect, all values of \( \theta \) are being used to estimate the effective value of \( \theta \) between the surface and level \( L \). From a physical point of view, however, it is better for the thickness between the surface and level \( L \) to depend only on the lowest-level value of \( \theta \). Arakawa and Suarez (1983) showed that under some conditions the form (9.177) can lead to large errors in the horizontal pressure-gradient.
force. We return to this point below.

It remains to specify the forms of $P_l$ and $\Theta_{l + \frac{1}{2}}$. Phillips (1974) suggested

$$P_l = \left( \frac{1}{1 + \kappa} \right) \left[ \delta(Pp) \right]_l,$$

(9.178)
on the grounds that this helps to give a good simulation of vertical wave propagation. The form of $\Theta_{l + \frac{1}{2}}$ can be chosen to permit conservation of some function of $\Theta$.

Arakawa and Suarez (1983) proposed a modified version of the scheme, in which (9.177) is replaced by

$$P_{\phi L} = \phi_S + A_{L + \frac{1}{2}} \theta L,$$

(9.179)

where $A_{L + \frac{1}{2}}$ is a nondimensional parameter discussed below. The point of (9.179) is that only $\theta_L$ influences the thickness between the surface and the middle of the bottom layer; the remaining values of $\theta$ do not enter. This makes the hydrostatic equation “local.” Arakawa and Suarez showed how this can be done with only minimal modifications to the derivation given above. The starting point is to replace (9.174) by

$$\phi_l - \phi_{l + 1} = c_p \left( A_{L + \frac{1}{2}} \theta_l + B_{l + \frac{1}{2}} \theta_{l + 1} \right),$$

(9.180)

and

where, again, $A_{L + \frac{1}{2}}$ and $B_{L + \frac{1}{2}}$ are to be determined. Substituting (9.180) and (9.171) into the identity

$$\phi_L - \phi_S = \sum_{l=1}^{L} \left[ \phi \delta \sigma - \delta (\sigma \phi) \right]_l - \sum_{l=1}^{L-1} \sigma_{l + \frac{1}{2}} (\phi_l - \phi_{l + 1}),$$

(9.181)

we obtain

$$\phi_L - \phi_S = \sum_{l=1}^{L} c_p \frac{P_l}{\partial \sigma} \delta P_l - \sum_{l=1}^{L-1} \sigma_{l + \frac{1}{2}} c_p \left( A_{L + \frac{1}{2}} \theta_l + B_{L + \frac{1}{2}} \theta_{l + 1} \right).$$

(9.182)

With the use of (9.146), we can write this as
\[ \phi_L - \phi_S = \sum_{l=1}^{L} c_p \theta_l \frac{\partial P_l}{\partial \sigma_l} \left( \delta \sigma_l \right) - \sum_{l=1}^{L-1} \sigma_l \frac{1}{2} \left( A_{l+\frac{1}{2}} + B_{l+\frac{1}{2}} \right) \theta_l. \] 

(9.183)

Every term on the right-hand-side of (9.183) involves a layer-center value of \( \theta \). We “collect terms” around individual values of \( \theta_l \) and force the coefficients to vanish for \( l < L \). This gives

\[ \pi \frac{\partial P_l}{\partial \sigma_l} \left( \delta \sigma_l \right) = \sigma_{l+\frac{1}{2}} + \sigma_{l-\frac{1}{2}}. \] 

(9.184)

With the use of (9.184), (9.183) reduces to

\[ \phi_L - \phi_S = \left[ \pi \frac{\partial P_L}{\partial \sigma} \left( \delta \sigma \right) - \sigma_{L-\frac{1}{2}} B_{L-\frac{1}{2}} \right] c_p \theta_L. \] 

(9.185)

which has the form of (9.179).

9.9 Summary and conclusions

The problem of representing the vertical structure of the atmosphere in numerical models is receiving a lot of attention at present. Among the most promising of the current approaches are those based on isentropic or quasi-isentropic coordinate systems. Similar methods are being used in ocean models.

At the same time, models are more commonly being extended through the stratosphere and beyond, while vertical resolutions are increasing; the era of hundred-layer models appears to be upon us.