Comment on the article "Vertical discretizations for compressible Euler equation atmospheric models giving optimal representation of normal modes" by Thuburn and Woollings

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In a recently published article [1], Thuburn and Woollings present a detailed analysis of the effect of the vertical discretization on the numerical representation of wave motion in the atmosphere. Among the schemes they analyzed, the ones that had the most accurate linear normal mode frequencies were not suited for preservation of discrete analogs of conservation properties. For example, the most straightforward way to achieve mass conservation is to predict density using the flux-form of the mass continuity equation, yet some of their optimal schemes did not involve such a direct prediction of mass. Here we augment their analysis by starting from a continuous form of the vertical momentum equation that involves the vertical derivative of the Exner function instead of pressure.[1] This improves the accuracy of the normal mode frequencies for the discretizations that may best preserve discrete analogs of conservation properties.

Thuburn and Woollings (hereafter TW) begin their article by discussing the continuous dry, non-hydrostatic, compressible Eulerian equations, in which, besides the three components of velocity, a minimum of two thermodynamic variables must be predicted. They present the possible pairs of prognostic thermodynamic variables and their corresponding prognostic equations within the framework of three different vertical coordinate systems: geometric height, isentropic and a terrain-following mass-based coordinate. Then they derive the analytical normal modes for wave motion in an atmosphere with an isothermal, hydrostatic basic state.

[1]We note here that immediately prior to submission of this article, we learned through personal correspondence with Dr. John Thuburn, the first author of [1], that he had independently performed an anlysis similar to ours. His work has recently been accepted for publication ([2]).

In the continuous system, the wave behavior is (of course) independent of the vertical coordinate and the set of prognostic variables. However, in the vertically discrete system, the normal modes depend on the vertical coordinate, the choice of prognostic thermodynamic variables, and the way these are staggered on the vertical grid. In their article TW analyze the discrete normal modes for all three vertical coordinate systems and an extensive set of prognostic thermodynamic variable pairs and staggerings. They categorize each configuration based on the accuracy of the represented wave frequency as well as on the existence of computational and/or unstable modes. Their results can serve as a guideline for choosing the prognostic variable set and the vertical grid on which to place them.

In the rating system used by TW, which is based on agreement with the analytical wave frequency, the top-rated configuration for each of the three vertical coordinate systems is not optimally suited for preserving discrete analogs of conservation properties. For example, none of these optimal configurations for the height and isentropic coordinates include mass as a predicted variable, and the ones for the isentropic and mass-based coordinates do not have height as a predicted variable. Therefore, satisfying the discrete analogs of mass and geopotential energy conservation in a fully non-linear model with these configurations may not be straightforward (see [3] and [4] for example). TW point out this issue in their paper and they do not discourage using configurations that have good conservation properties but have less accurate wave frequency representation.

Is there a way to make the configurations with good conservation properties have optimal normal mode frequency representation as well? We performed a numerical eigenmode analysis of these configurations using a form of the vertical pressure-gradient term that uses the Exner function instead of pressure. As mentioned above, this results in a more accurate representation of the normal mode frequencies. In the following we contrast the vertical momentum equations we used with those used by TW for each of the three vertical coordinate systems and present our results. The configurations we focus on are only those that include mass, geopotential and potential temperature as the prognostic thermodynamic variables, as these have the best conservation properties in terms of mass, geopotential and thermodynamic energy.

Height coordinate:

The height-coordinate configuration with good conservation properties has mass predicted at the levels containing the horizontal velocity. It is denoted by TW with the notation ($w\theta$, $uv\rho$) which lists the prognostic variable set. The variables preceding the comma are at layer edges, while those following the comma are at layer centers as shown in Figure 1. Here w is the vertical velocity, θ the potential temperature, u and v are the horizontal velocity components, and ρ is density.

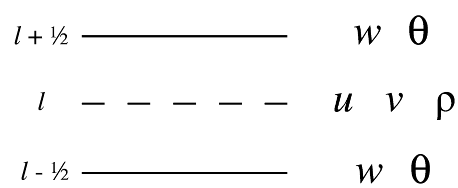

FIG. 1: Staggering of the prognostic variables on the vertical z-coordinate grid.

The form of the continuous non-linear vertical momentum equation that TW consider is

$$\frac{Dw}{Dt} = -\frac{1}{\rho}p_z - g. \tag{1}$$

The corresponding linearized form is

$$w_t = -\frac{1}{\rho^{(r)}}\left(p_z + g\rho\right). \tag{2}$$

We use the same notation as TW, where subscripts z and t indicate partial derivatives, the superscript (r) represents the isothermal, hydrostatic, resting basic state and the variables represent perturbation quantities. Also p is pressure, g is gravity and $\frac{D}{Dt}$ is the material time derivative. The vertically discrete form of equation (2) is

$$w_{t l+1/2} = -\frac{1}{\rho^{(r)}{}_{l+1/2}}\left[\frac{(\delta p)_{l+1/2}}{(\delta z)_{l+1/2}} + g\frac{1}{2}\left(\rho_{l+1} + \rho_l\right)\right], \tag{3}$$

where the subscript l is the integer layer-center index. Variables with half-integer indices are located at layer edges. The vertical difference of the variable A is defined as

$$(\delta A)_{l+1/2} \equiv A_{l+1} - A_l. \tag{4}$$

Note that the buoyancy term involves density averaged to layer centers.

The alternate form of the non-linear vertical momentum equation that we used is

$$\frac{Dw}{Dt} = -\theta\Pi_z - g, \tag{5}$$

where Π is the Exner function defined by $\Pi \equiv c_p(\frac{p}{p_0})^\kappa$. The resulting linearized equation is

$$w_t = -\theta^{(r)}\Pi_z + \frac{g}{\theta^{(r)}}\theta. \tag{6}$$

The vertically discrete form of (6) is

$$(w_t)_{l+1/2} = -\theta^{(r)}{}_{l+1/2}\frac{(\delta\Pi)_{l+1/2}}{(\delta z)_{l+1/2}} + \frac{g}{\theta^{(r)}{}_{l+1/2}}\theta_{l+1/2}. \tag{7}$$

Now there is no averaging of prognostic variables involved because the buoyancy term is written in terms of θ, which is located at the same level as w. There are no other changes to the system of equations except for the addition of a diagnostic relation between the Exner function and pressure. Since this relation does not involve averaging, because Π and p are at the same level, we have effectively reduced the number of averages in the system by one. The resulting discrete normal mode frequencies are closer to the analytical wave frequencies and match those of the "optimal" category of TW. We believe the improved accuracy is due to the fact that fewer variables are averaged. This supports TW's argument to that effect.

The eigenmode problem analyzed is the solution of the normal modes of an atmosphere with an isothermal, hydrostatic basic state at rest. The vertical domain has depth D (= 10 km) and is bounded on top and bottom by rigid, constant-height surfaces. The equations are continuous in time and in the horizontal dimension and the horizontal domain is periodic. Waves with a horizontal wavelength of 1000 km are considered. Figure 22 reproduced from TW's paper, is a representative dispersion relation for the schemes they classify as Category 1 configurations. These have distinctly more accurate normal mode frequencies than other configurations. The discrete normal mode frequencies we obtained by using equation (7) match this figure. The frequency of the westward propagating acoustic (highest frequency), gravity, and Rossby (lowest frequency) waves are shown as a function of vertical wave mode n which is related to the vertical wave number m by the relation $m = n\pi/D$. The numerical frequencies are indicated by the crosses, and the analytical frequencies by the diamonds. Each vertical wave mode has n - 1 internal nodes. There are 20 levels, therefore the highest mode that can be represented is $n = 19$, which has a wavelength slightly longer than $2(\delta z)$.

The $4(\delta z)$ mode is given by $n = 10$. The two $n = 0$ modes are the external acoustic (Lamb wave) and Rossby modes. There is generally good agreement between the numerical and analytical frequencies. The error is larger for modes with higher wave numbers.

In contrast to the "Category 1" results we obtained using the "Exner function form" of the vertical momentum equation, TW obtained the dispersion relation shown in Figure 3 with their ($w\theta$, $uv\rho$) configuration through the use of equation (3). This figure is representative of schemes that TW placed in Category 2a, in which no zero-frequency computational modes exist but there is a substantial underestimate of the Rossby mode frequencies at the higher vertical wave numbers. So we were able to upgrade the category with good conservation properties by using the alternate form of the vertical momentum equation. This reclassification is shown in Table I, which summarizes the results of using the "Exner function form" of the vertical momentum equation with the configurations best suited for satisfying conservation properties, i.e., the mass and/or height-predicting configurations. The schemes we analyzed have the subscript "Π".

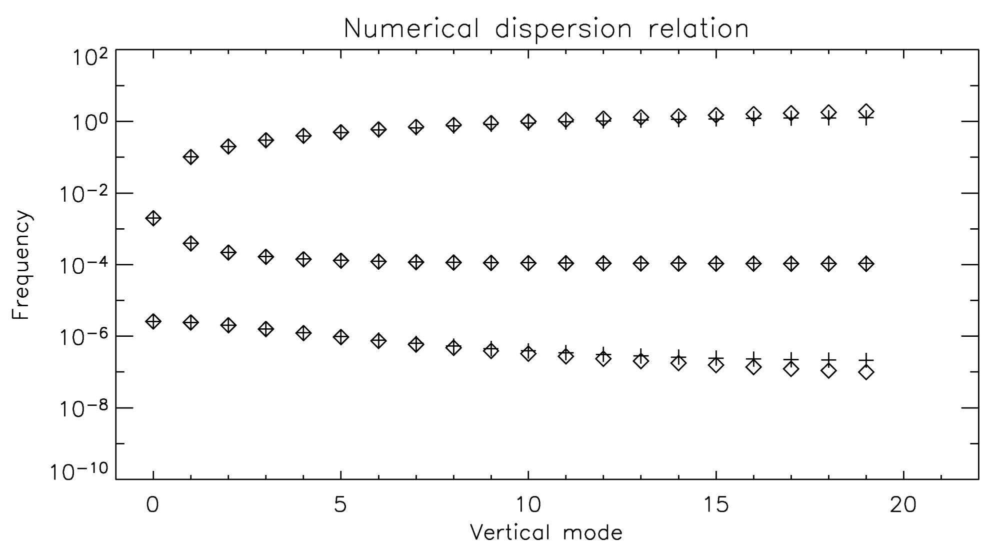

FIG. 2: Representative dispersion relation (frequency in s^{-1}) for Category 1 "optimal" configurations reproduced from [1]. Crosses indicate numerical eigenmode frequencies and diamonds indicate analytical frequencies.

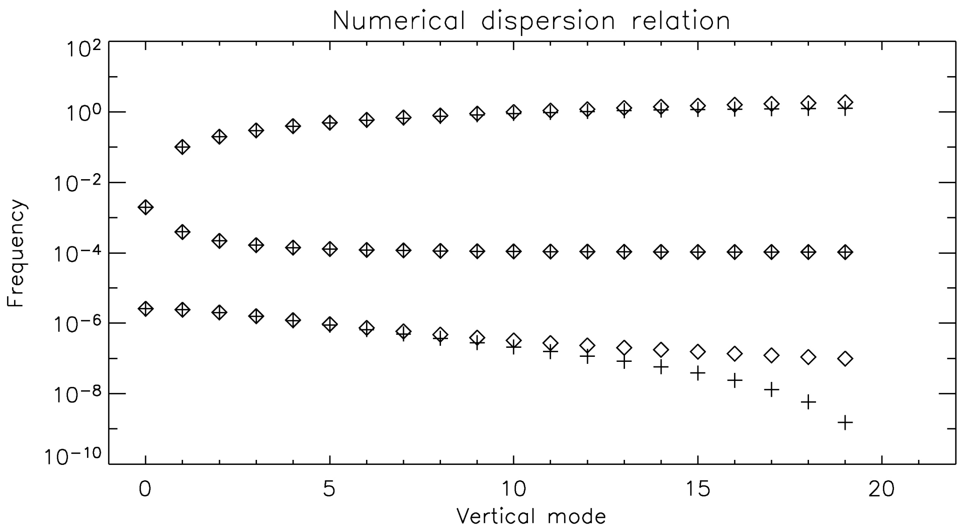

FIG. 3: Representative dispersion relation (frequency in s^{-1}) for Category 2a "slow Rossby mode" configurations reproduced from [1].

TABLE I: Reclassification of selected mass- and/or height-predicting configurations resulting from modification of the vertical momentum equation. These configurations have the superscript 'Π' to denote use of the "Exner function form" of the vertical momentum equation. The other configurations are as in [1].

	Height coordinate	*Isentropic coordinate*	*T*errain-following mass-based coordina
Category 1 (optimal category)	$(w\theta, uv\rho)^{\Pi}$		$(w\theta z, uv)^{\Pi}$
Category 2a (slow Rossby modes)	$(w\theta, uv\rho)$	$(wz, uv\sigma)^{\Pi}$	$(w\theta z, uv)$
Category 2b (fast Rossby modes)		$(wz, uv\sigma)^{p}$	

Isentropic coordinate:

For the case of isentropic coordinates, we analyzed the $(wz, uv\sigma)^p$ configuration whose vertical grid staggering is shown in Figure 4.

The form of the continuous non-linear vertical momentum equation in θ-coordinates that Thuburn and Woollings consider for the mass and height predicting configurations is

$$\frac{Dw}{Dt} = -\frac{1}{\sigma} p_{\theta} - g, \tag{8}$$

where σ is the pseudo-density defined by $\sigma \equiv \rho \frac{\partial z}{\partial \theta}$ and z is the geopotential height on

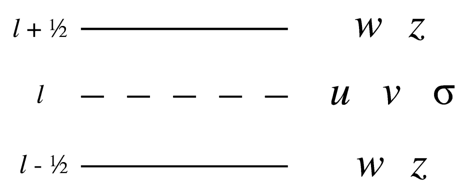

FIG. 4: Staggering of the prognostic variables on the vertical θ-coordinate grid.

θ-coordinate surfaces. The resulting linearized form is

$$w_t = -\frac{1}{\sigma^{(r)}}\left(p_\theta + g\sigma\right), \tag{9}$$

where the variables represent perturbation quantities. The vertically discrete form of (9) is

$$(w_t)_{l+1/2} = -\frac{1}{{\sigma^{(r)}}_{l+1/2}}\left[\frac{(\delta p)_{l+1/2}}{(\delta\theta)_{l+1/2}} + g\frac{1}{2}\left(\sigma_{l+1} + \sigma_l\right)\right]. \tag{10}$$

This equation involves averaging the pseudo-density to layer edges in the buoyancy term. The diagnostic equation for density ρ is given by

$$\frac{\sigma_l}{{\sigma^{(r)}}_l} = \frac{\rho_l}{{\rho^{(r)}}_l} + \frac{1}{{z_\theta}^{(r)}{}_l}\frac{(\delta z)_l}{(\delta\theta)_l}. \tag{11}$$

The result that TW obtain for this configuration is shown in Figure 5. This dispersion relation is representative of Category 2b schemes in which the frequency of the Rossby modes for high vertical wave numbers is substantially overestimated.

The alternate form of the continuous non-linear vertical momentum equation that we consider is

$$\frac{Dw}{Dt} = -\frac{\rho}{\sigma}\theta\Pi_\theta - g. \tag{12}$$

In our eigenmode analysis we linearize the discrete form of equation (12). This gives a different result from discretizing the equation in its linear form, and we believe it provides a more accurate representation of the linearized characteristics of an actual non-linear model. The resulting linearized discrete equation is

$$(w_t)_{l+1/2} = -\theta_{l+1/2}\frac{(\delta\Pi)_{l+1/2}}{(\delta z^{(r)})_{l+1/2}} - g\frac{1}{2}\left[\frac{(\delta z)_{l+1}}{(\delta z^{(r)})_{l+1}} + \frac{(\delta z)_l}{(\delta z^{(r)})_l}\right]. \tag{13}$$

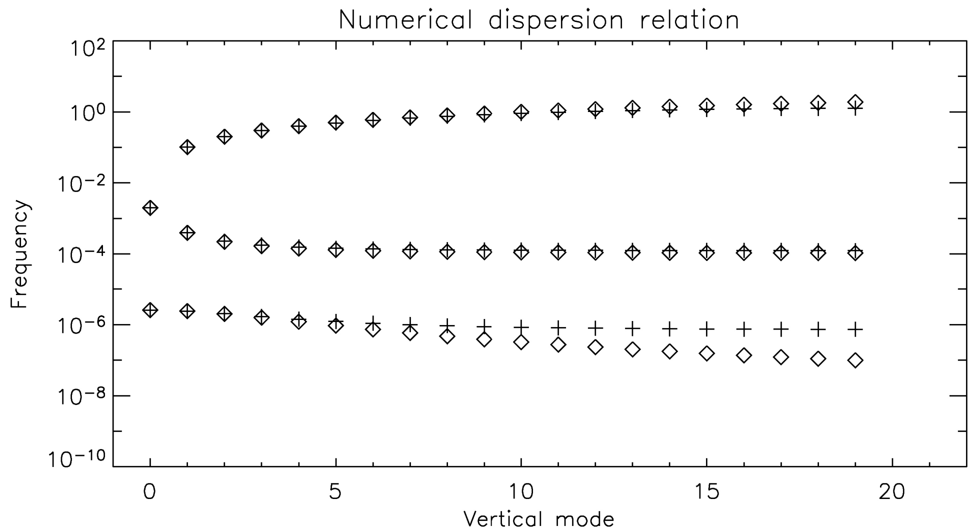

FIG. 5: Representative dispersion relation (frequency in s^{-1}) for Category 2b "fast Rossby mode" configurations reproduced from [1].

Note that there is still one averaged quantity, i.e., the average of the difference $(\delta z)_l/(\delta z^{(r)})_l$, which is equivalent to the "coarse" difference of z across $2(\delta z^{(r)})_l$.

The dispersion relation we obtained using this "Exner function form" of the vertical momentum equation is the same as that for the Category 2a schemes shown in Figure 3. So the effect of using this equation in the eigenmode analysis is to alter the accuracy of the wave frequency from a high bias to a low bias (see Table I). For this case the effect is neutral, i.e., there is not a definite improvement in accuracy.

Terrain-following mass-based coordinate:

The vertical staggering of the ($w\theta z$, uv) configuration in the terrain-following mass-based coordinate system is shown in Figure 6.

The non-linear form of the vertical momentum equation that TW linearize is

$$\frac{Dw}{Dt} = -\frac{1}{\sigma}\frac{\partial p}{\partial \eta} - g, \tag{14}$$

where η is the mass-based coordinate defined at the geometric height z as the ratio of the mass of air per unit area above that level to the total mass of air per unit area in the entire

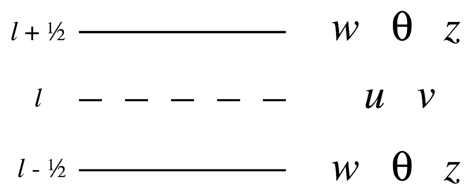

FIG. 6: Staggering of the prognostic variables on the vertical η-coordinate grid.

column. This is expressed as

$$\eta \equiv \frac{\int_z^{z_T} \rho dz}{\int_{z_S}^{z_T} \rho dz}, \tag{15}$$

where z_T and z_S are the model top and surface height respectively. The pseudo-density, defined as $\sigma \equiv \rho \frac{\partial z}{\partial \eta}$, is the total mass per unit area of the atmospheric column above a given point on the surface, i.e.,

$$\sigma = \int_{z_S}^{z_T} \rho dz. \tag{16}$$

When linearized, equation (14) becomes

$$w_t = -\frac{1}{\sigma^{(r)}} \left(p_\eta + g\sigma \right). \tag{17}$$

The vertically discrete form is then

$$(w_t)_{l+1/2} = -\frac{1}{\sigma^{(r)}} [\frac{(\delta p)_{l+1/2}}{(\delta \eta)_{l+1/2}} + g\sigma]. \tag{18}$$

Note that the pseudo-density σ is independent of height and therefore does not carry a level index. This form of the vertical momentum equation involves no averaged variables.

The "Exner function form" of the non-linear vertical momentum equation is

$$\frac{Dw}{Dt} = -\frac{\rho}{\sigma} \theta \frac{\partial \Pi}{\partial \eta} - g. \tag{19}$$

We linearize the discrete form of (19), which gives

$$(w_t)_{l+1/2} = -{\theta^{(r)}}_{l+1/2} \frac{(\delta \Pi)_{l+1/2}}{(\delta z^{(r)})_{l+1/2}} + g \frac{\theta_{l+1/2}}{{\theta^{(r)}}_{l+1/2}} - g \frac{1}{2} [\frac{(\delta z)_{l+1}}{(\delta z^{(r)})_{l+1}} + \frac{(\delta z)_l}{(\delta z^{(r)})_l}]. \tag{20}$$

Comparing equations (18) and (20), we see that we have introduced an averaged quantity. According to the argument about the relationship between accuracy and the number of averaged variables in the system we should expect the normal mode frequencies to be less accurate. Instead the accuracy improves and this configuration moves to the "optimal" category as shown in Table I. The reason for this is not clear.

Conclusion

We have demonstrated that it is possible to optimally represent normal mode frequencies with vertical discretizations that are best suited for preserving conservation properties. This is achieved by using the Exner function instead of pressure in the vertical pressure-gradient term of the vertical momentum equation. We analyzed the effect of this modification in the three vertical coordinates that TW considered. In height coordinates, the improved accuracy of the normal modes appears to be due to the reduction in the number of averaged variables in the system of equations. For the mass-based coordinates, however, the accuracy improves despite the introduction of an additional averaged quantity. For the isentropic coordinate, the "optimal" category is not achieved, although it can be argued that there is improvement in moving from Category 2b to 2a. Figures 3 and 5 show that the error in the Rossby mode frequency mainly occurs in the higher vertical modes – in the case of the "slow Rossby mode" category (Category 2a) the error becomes large for n greater than about 10 ($4(\delta z)$ and shorter waves), and for the "fast Rossby mode" category (Category 2b) waves with n greater than about 5 ($8(\delta z)$ and shorter) are affected. Waves whose wavelengths are shorter than $4(\delta z)$ are generally not well represented in models so the error in the frequencies for such waves is probably not important. Since the "slow Rossby mode" category more accurately represents frequencies of waves in the $4(\delta z)$ and larger wavelengths than the "fast Rossby mode" category, it is the better category. In fact it is almost as accurate as the "optimal" category for this range of wavelengths.

[1] J. Thuburn and T. J. Woollings, Vertical discretizations for compressible Euler equation atmospheric models giving optimal representation of normal modes, J. Comput. Phys., **203**, 386-404, (2005).

[2] J. Thuburn, Vertical discretizations giving optimal representation of normal modes: Sensitivity to the form of the pressure gradient term, Q. J. R. Meteorol. Soc., (in press).

[3] D. W. Byun, Dynamically consistent formulations in meteorological and air quality models for multiscale atmospheric studies. Part I: Governing equations in a generalized coordinate system, J. Atmos. Sci., **56**, 3789-3807, (1999).

[4] K. E. Taylor, A vertical finite-difference scheme for hydrostatic and nonhydrostatic equations, Mon. Wea. Rev., **112**, 1398-1402, (1984).