# Potential Vorticity as Meridional Coordinate

CRISTIANA STAN* AND DAVID A. RANDALL

*Department of Atmospheric Science, Colorado State University, Fort Collins, Colorado*

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ABSTRACT

The dynamical equations of atmospheric flow are written using potential vorticity as the meridional coordinate and potential temperature as the vertical coordinate. Within this system, the atmosphere is divided into undulating tubes bounded by isentropic and constant potential vorticity surfaces, and, under adiabatic and frictionless conditions, the air moves through the tubes without penetrating through the walls.

A model that uses this system of coordinates incorporates as built-in both the dry convective adjustment and barotropic adjustment processes, which prevent the folding of isentropic and potential vorticity surfaces but crudely represent the effects of interactions with the mean flow.

The Eliassen–Palm flux vector in this frame of reference yields a new interpretation of the eddy momentum transport. Eddy momentum exchange is through the form drag created by pressure forces exerted on the potential vorticity–potential temperature tubes.

## 1. Introduction

The classical system of coordinates consisting of longitude, latitude, and height has been altered, mainly in the vertical, to facilitate the analysis and simulation of atmospheric flow. Kasahara (1974) gives a complete mathematical description of the quasi-static dynamical equations using height, pressure, $\sigma$, and potential temperature, and mentions the advantages and disadvantages of each of these coordinates. For example, the pressure coordinate has the advantages of a linear continuity equation and horizontal pressure-gradient term, but it has difficulties with the lower boundary conditions. Using $\sigma$ as vertical coordinate avoids this problem, since in $\sigma$ coordinates the earth's surface is a coordinate surface. Although this solves the difficulties posed by the intersection of the coordinate surfaces with the topography, the representation of the horizontal pressure-gradient term in $\sigma$ coordinates is problematic in the vicinity of steep topography (e.g., Kasahara 1974; Konor and Arakawa 1997). The isentropic coordinates has difficulties at the earth's surface, but it offers a simple representation of the horizontal pressure-gradient term, and vertical advection is solely due to heating. In the absence of heating, isentropic surfaces are material surfaces. This ensures that particles move along the coordinate surfaces. In the case of height, pressure, and $\sigma$ coordinates the vertical advection is more often nonzero and fluid particles cross the coordinate surfaces whether heating occurs or not. A detailed analysis of the advantages and disadvantages of using potential temperature as vertical coordinate is given by Hsu and Arakawa (1990).

In this paper, we propose a system of coordinates (PVPT coordinates) that consists of longitude, potential vorticity (PV), and potential temperature (PT). The general definition of potential vorticity, as introduced by Ertel (1942) for the adiabatic atmosphere is

$$q = \frac{1}{\rho}\boldsymbol{\zeta}_a \cdot \nabla\theta, \tag{1}$$

where $\rho$ is the fluid density, $\boldsymbol{\zeta}_a$ is the absolute vorticity vector, and $\theta$ is the potential temperature. When expressed in isentropic coordinates with the quasi-static approximation, (1) simplifies to

$$q = \frac{f + \zeta_\theta}{-g^{-1}(\partial p/\partial\theta)}, \tag{2}$$

* Current affiliation: Center for Ocean–Land–Atmosphere Studies, Calverton, Maryland.

*Corresponding author address:* Cristiana Stan, Center for Ocean–Land–Atmosphere Studies, Suite 302, Calverton, MD 20705-3106.
E-mail: stan@cola.iges.org

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where $\zeta_\theta$ is the vertical component of the relative vorticity on the isentropic surface. Thus PV is generally positive in the Northern Hemisphere, negative in the Southern Hemisphere, and close to zero near the equator.

There are three ways in which the use of PV as meridional coordinate is analogous to the use of potential temperature as vertical coordinate: (i) PV is materially conserved under adiabatic, frictionless processes, just as potential temperature is materially conserved under adiabatic processes. Consequently, the use of PV as meridional coordinate eliminates meridional advection under adiabatic frictionless processes, just as the use of potential temperature as the vertical coordinate eliminates vertical advection under adiabatic processes; (ii) PV normally (but not always) increases toward the north, just as potential temperature normally (but not always) increases upward. A northward decrease of PV is an indicator of barotropic–baroclinic instability (Charney and Stern 1962), just as an upward decrease of potential temperature is an indicator of convective instability; (iii) PV is not constant at the north and south poles, just as potential temperature is not constant along the earth's surface. There are complications in the formulation of the polar boundary conditions with the PV coordinate, just as there are complications in the formulation of the lower boundary condition with the potential temperature coordinate. Another major disadvantage of the PVPT system is that it is unable to represent large-scale circulation features that involve PV-gradient reversals, such as baroclinic wave life cycles, blocking events, and cutoff cyclones.

This is not the first time that an alternative meridional coordinate has been proposed. For example, Hoskins (1975) replaced the geographical latitude and longitude with the semigeostrophic coordinates that specify the positions of particles as they would move with the geostrophic velocity. Shutts (1980) used the angular momentum as meridional coordinate and investigated its usefulness for the study of the mean merdional circulation of the atmosphere. Schubert and Magnusdottir (1994) generalized the semigeostrophic coordinates to the so-called vorticity coordinates. The vorticity coordinates have the property that their Jacobian with respect to the geographical latitude and longitude is the vertical component of the absolute vorticity vector. The vorticity coordinates are an appropriate frame of reference to investigate the zonally symmetric balanced flows. Nakamura (1995, 1996) defined the meridional coordinate as the mass/area enclosed by the contour of a conservative variable on an isentropic surface. Allen and Nakamura (2003) employed the PV area equivalent latitude (PVEL) for tracer analysis.

Obukhov (1964) showed that surfaces of PV and potential temperature divide the atmosphere into undulating tubes within which the air particles share the same PV and potential temperature. For adiabatic and frictionless conditions, the walls of the tubes are impermeable, and the dynamics of the flow can be studied from a Lagrangian perspective by following the displacement of the tubes. Although gravity waves are invisible in the PV field, because the vorticity vector of the gravity waves is tangential to the isentropic surfaces (Eliassen 1987), a model that uses PV as its meridional coordinate can simulate gravity waves.

Arakawa (1957) and Eliassen and Palm (1961) showed that the eddy momentum flux and the eddy heat flux do not act independently in modifying the mean state of the atmosphere. Their effects can be combined into a vectorial quantity with components in the latitude–height plane known as the Eliassen–Palm (EP) flux. The EP flux is a powerful tool that provides information about the center of the action of the eddies and the divergence of the EP flux can be interpreted as the forcing of the mean merdional circulation (e.g., Charney and Drazin 1961; Dickinson 1969; Andrews and McIntyre 1976, 1978; Boyd 1976; Andrews 1983). Although the EP flux vector is defined independent of the system of coordinates, when projected on different systems of coordinates its components capture different aspects of the eddy–mean flow interaction. For example, in pressure coordinates the eddies are seen to drive a mean meridional circulation with a thermally indirect cell located in middle latitudes, whereas in isentropic coordinates the mean meridional circulation is dominated by a thermally direct cell that extends all the way from the equator to the pole (Townsend and Johnson 1985; Johnson 1989).

Despite the fact that a PV perspective on the general circulation of the atmosphere brings powerful insights and also despite the fact that isentropic analyses have been extensively carried out, the advantages of the PVPT coordinates were not employed in 3D modeling of the atmosphere. We will also show that PVPT coordinates are useful as a diagnostic tool for understanding the physical processes governing the atmospheric flow and for the interpretation of observations.

A PVPT-based model is presented in section 2. The section begins with a brief description of the basic equations. Diabatic and frictional effects are included. We discuss the implications of the requirement that the PV changes monotonically with latitude. The theory developed in section 2 is illustrated in section 3 in which a shallow-water model that uses potential vorticity as meridional coordinates is studied. In section 4 a new form of the EP flux vector is derived by considering the zonally averaged angular momentum along a PV contour on an isentropic surface. Finally, the results of this study are summarized in section 5.

## 2. The primitive equations in PVPT coordinates

### a. Starting equations

Before deriving the governing equations in the PVPT system of coordinates, we review the basic equations in isentropic coordinates, with an emphasis on the properties that make them useful for our purpose. The primitive equations in isentropic coordinates expressing horizontal momentum balance, mass conservation, thermodynamic balance between the Lagrangian change of potential temperature and diabatic sources, and hydrostatic balance in the vertical can be written as

$$\frac{DU}{Dt} - 2\Omega\mu V + \frac{1}{a}\left(\frac{\partial M}{\partial \lambda}\right)_{\mu,\theta} = X_\lambda, \quad (3)$$

$$\frac{DV}{Dt} + 2\mu\left(\Omega U + \frac{K}{a}\right) + \frac{1-\mu^2}{a}\left(\frac{\partial M}{\partial \mu}\right)_{\lambda,\theta} = X_\mu, \quad (4)$$

$$\frac{D\sigma}{Dt} + \frac{1}{a}\left\{\frac{\partial}{\partial \lambda}\left(\frac{\sigma U}{1-\mu^2}\right)\right\}_{\mu,\theta} + \frac{1}{a}\left\{\frac{\partial}{\partial \mu}(\sigma V)\right\}_{\lambda,\theta} + \left\{\frac{\partial}{\partial \theta}(\sigma\dot{\theta})\right\}_{\lambda,\mu} = 0, \quad (5)$$

$$\left(\frac{\partial M}{\partial \theta}\right)_{\lambda,\mu} = \Pi, \quad (6)$$

where $U = u\cos\varphi$ and $V = \upsilon\cos\varphi$ are the modified components of the horizontal velocity vector, $\mu = \sin\varphi$, and $\lambda$ is longitude. Subscripts denote that derivatives are calculated at fixed longitude ($\lambda$), latitude ($\mu$), and/or potential temperature ($\theta$). The quantity $M$ represents the Montgomery streamfunction and is defined by

$$M = \theta\Pi(p) + gz, \quad (7)$$

where $\Pi(p)$ is the Exner function, which depends on pressure only. The terms $X_\lambda$ and $X_\mu$ are the horizontal components of friction or other nonconservative mechanical forcing. The horizontal kinetic energy per unit mass is denoted by

$$K = \frac{1}{2}\left(\frac{U^2 + V^2}{1-\mu^2}\right).$$

The isentropic-mass

$$\sigma \equiv -\frac{1}{g}\left(\frac{\partial p}{\partial \theta}\right)_{\lambda,\mu}$$

is the pseudodensity in the ($\lambda$, $\mu$, $\theta$) space. The Lagrangian time rate of change is given by

$$\frac{D}{Dt} = \left(\frac{\partial}{\partial t}\right)_{\lambda,\mu,\theta} + \frac{U}{a(1-\mu^2)}\left(\frac{\partial}{\partial \lambda}\right)_{\mu,\theta} + \frac{V}{a}\left(\frac{\partial}{\partial \mu}\right)_{\lambda,\theta} + \frac{\dot{\theta}}{a}\left(\frac{\partial}{\partial \theta}\right)_{\lambda,\mu}. \quad (8)$$

The PV principle associated with the isentropic system described by (3)–(6) is

$$\frac{Dq}{Dt} = \dot{q}, \quad (9)$$

where the PV is given by

$$q = \frac{1}{\sigma}\left\{2\Omega\mu + \frac{1}{a(1-\mu^2)}\left(\frac{\partial V}{\partial \lambda}\right)_{\mu,\theta} - \frac{1}{a}\left(\frac{\partial U}{\partial \mu}\right)_{\lambda,\theta}\right\}, \quad (10)$$

and $\dot{q}$ contains effects due to diabatic and frictional terms in the momentum equation. In the absence of diabatic processes and friction, $\dot{q} = 0$, PV is conserved following a particle, and surfaces of constant PV act as material surfaces.

### b. Transforming to PVPT coordinates

Before we actually perform the transformation, it is useful to discuss the geometry of the PVPT system. Since we want the newly created system to be orthogonal, we chose its unit vectors to remain parallel to the unit vectors of a spherical system of coordinates, as schematically shown in Fig. 1. Therefore, in the PVPT system, the unit vectors are not orthogonal to the isosurfaces of the coordinates, and the transformation of coordinates has to reflect this. This requirement ensures that the mass contained in the unit volume in the old system remains the same in the unit of volume expressed in the new system.

To quantify the tilting of the PV coordinate from the geographical latitude $\mu$, we adopt Kushner and Held's (1999) definition of "potential-vorticity thickness"; that is,

$$h = \left(\frac{\partial \mu}{\partial q}\right)_{\lambda,\theta}. \quad (11)$$

We require that $h$ be positive and finite, which is equivalent to the condition that the PV is a monotonic function of $\mu$. Using the definition of PV thickness, the gradient components of any arbitrary function $A(\lambda, \mu, \theta, t)$ transform to the PVPT system as

$$\left(\frac{\partial A}{\partial \alpha_i}\right)_{\mu,\alpha_j} = \left(\frac{\partial A}{\partial \alpha_i}\right)_{q,\alpha_j} - \frac{1}{h}\left(\frac{\partial A}{\partial q}\right)_{\alpha_i}\left(\frac{\partial \mu}{\partial \alpha_i}\right)_{q,\alpha_j}, \quad (12)$$

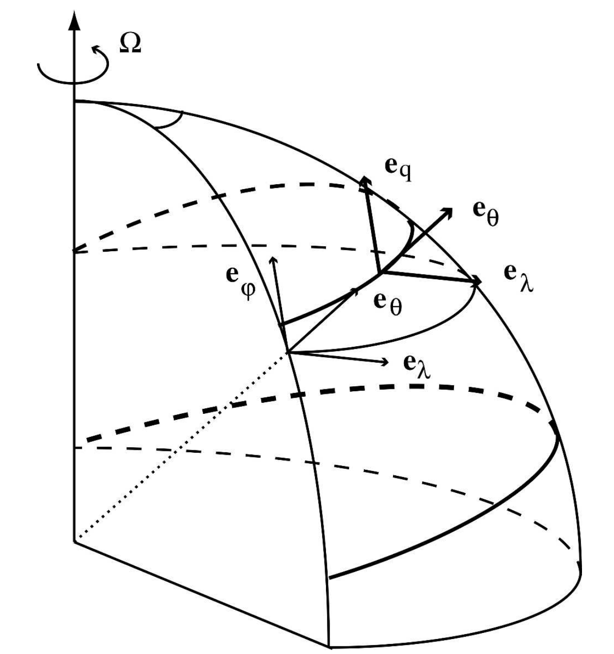


FIG. 1. Illustration of the unit vectors of a local spherical system of coordinates ($\mathbf{e}_\lambda$, $\mathbf{e}_\varphi$, $\mathbf{e}_r$) using thin line and the unit vectors of the PVPT system of coordinates ($\mathbf{e}_\lambda$, $\mathbf{e}_q$, $\mathbf{e}_\theta$) using thick line.

where $\alpha_i$, $\alpha_j$ can be any variables among ($\lambda$, $\theta$, $t$), and

$$\left(\frac{\partial A}{\partial \mu}\right)_{\lambda,\theta} = \frac{1}{h}\left(\frac{\partial A}{\partial q}\right)_{\lambda,\theta}. \tag{13}$$

Using (12) and (13), the PV principle (9) transforms to an equation that predicts the geographical latitude of a PV contour; that is,

$$\left(\frac{\partial \mu}{\partial t}\right)_{q,\theta} + \frac{U}{a(1-\mu^2)}\left(\frac{\partial \mu}{\partial \lambda}\right)_{q,\theta} + \frac{\dot{q}}{a}h + \dot{\theta}\left(\frac{\partial \mu}{\partial \theta}\right)_{\lambda,q} = \frac{V}{a}. \tag{14}$$

From (14) it is evident that ($U$, $\dot{q}$, $\dot{\theta}$) are the components of the velocity vector expressed in the PVPT coordinates.

The meridional component of the horizontal velocity in the isentropic system, which appears on the right-hand side of (14), needs careful consideration. In the PVPT system, $V$ represents a measure of the rate of meridional displacement of the PVPT tubes. A novelty of this system is that the meridional advection is due to physical processes associated with $\dot{q}$ rather than dynamics. The physical processes may be specified through parameterizations.

The continuity equation in the PVPT system becomes

$$\left(\frac{\partial m}{\partial t}\right)_{\lambda,q,\theta} + \frac{1}{a}\left\{\frac{\partial}{\partial \lambda}\left(\frac{mU}{1-\mu^2}\right)\right\}_{q,\theta} + \frac{1}{a}\left\{\frac{\partial}{\partial q}(m\dot{q})\right\}_{\lambda,\theta} + \left\{\frac{\partial}{\partial \theta}(m\dot{\theta})\right\}_{\lambda,q} = 0, \tag{15}$$

where $m = \sigma h$ denotes the PVPT mass. It can be interpreted as the pseudodensity in the ($\lambda$, $q$, $\theta$) space and is defined as

$$m \equiv -\frac{h}{g}\left\{\left(\frac{\partial p}{\partial \theta}\right)_{\lambda,q} - \frac{1}{h}\left(\frac{\partial p}{\partial q}\right)_{\lambda,\theta}\left(\frac{\partial \mu}{\partial \theta}\right)_{\lambda,q}\right\}. \tag{16}$$

For adiabatic and frictionless conditions, the continuity equation has a very simple form; it says that the PVPT mass changes only due to zonal advection along the PVPT tubes bounded by impermeable walls. Under these circumstances a PVPT tube can inflate in some regions and deflate in others in order to allow rearrangement of the air in the tube. Note that the metric quantity $(1 - \mu^2)$ appears in both forms (5) and (15), but in (15) it is not independent of $\lambda$, while in (5) it is.

The hydrostatic equation describing the vertical balance in the PVPT frame of reference is

$$\left(\frac{\partial M}{\partial \theta}\right)_{\lambda,q} - \frac{1}{h}\left(\frac{\partial M}{\partial q}\right)_{\lambda,\theta}\left(\frac{\partial \mu}{\partial \lambda}\right)_{q,\theta} = \Pi. \tag{17}$$

The hydrostatic equilibrium expressed by (17) is different from the usual hydrostatic conditions, and the difference occurs in the presence of meridional component of the pressure gradient force. The second term on the left-hand side of (17) can be interpreted as a correction to account for the fact that $(\partial M/\partial \theta)_{\lambda,q}$ is evaluated at constant $q$, but not necessarily constant latitude. In other words, it accounts for the possible tilting of the $q$ surfaces in the meridional direction. This tilting only changes in the answer if $M$ varies with $q$ at fixed longitude and $\theta$ and if the latitude on the $q$ surface varies with longitude. Equation (17) can also be expanded in terms of the geometrical height of the isentropic surface, $z$:

$$\theta\left(\frac{\partial \Pi}{\partial \theta}\right)_{\lambda,q} + g\left(\frac{\partial z}{\partial \theta}\right)_{\lambda,q} - \frac{\theta}{h}\left(\frac{\partial \Pi}{\partial q}\right)_{\lambda,\theta}\left(\frac{\partial \mu}{\partial \theta}\right)_{\lambda,q} - \frac{g}{h}\left(\frac{\partial z}{\partial q}\right)_{\lambda,\theta}\left(\frac{\partial \mu}{\partial \theta}\right)_{\lambda,q} = 0. \tag{18}$$

This form is useful for the analysis of the horizontal pressure-gradient term. More details are given by Stan (2005).

In the PVPT system, the zonal momentum equation becomes

$$\frac{DU}{Dt} - 2\Omega\mu V + \frac{1}{a}\left\{\left(\frac{\partial M}{\partial \lambda}\right)_{q,\theta} - \frac{1}{h}\left(\frac{\partial M}{\partial q}\right)_{\lambda,\theta}\left(\frac{\partial \mu}{\partial \lambda}\right)_{q,\theta}\right\} = X_\lambda, \quad (19)$$

where the Lagrangian derivative is given by

$$\frac{D}{Dt} = \left(\frac{\partial}{\partial t}\right)_{\lambda,q,\theta} + \frac{U}{a(1-\mu^2)}\left(\frac{\partial}{\partial \lambda}\right)_{q,\theta} + \frac{\dot{q}}{a}\left(\frac{\partial}{\partial q}\right)_{\lambda,\theta} + \dot{\theta}\left(\frac{\partial}{\partial \theta}\right)_{\lambda,q}. \quad (20)$$

By comparing (20) with (8) we notice the PVPT coordinates offer a simpler representation of the material derivative in adiabatic and frictionless conditions.

To close the PVPT system, we also need to transform (4), which becomes

$$\frac{DV}{Dt} + 2\mu\left(\Omega U + \frac{K}{a}\right) + \frac{1-\mu^2}{ah}\left(\frac{\partial M}{\partial q}\right)_{\lambda,\theta} = X_q. \quad (21)$$

The zonal equation of motion (19), describes the acceleration of a particle of fluid that moves in the zonal direction through the PVPT tubes, whereas (21) gives the meridional acceleration of the PVPT tube.

Unlike an isentropic model, a model that uses PV as meridional coordinate has four prediction equations, that is, (14), (15), (19), and (21). One of them, (14), describes the motion of the coordinate itself. Applying the vorticity coordinates to the shallow-water model, Schubert and Magnusdottir (1994) similarly obtained a model with four prognostic equations instead of three as in the starting model. In a numerical model based on (14)–(21), the variables that need to be initialized at $t = 0$ are $U$, $V$; the geographical latitude of the PV contour, $\mu$; and the pressure necessary to calculate the Exner function or the PVPT mass, $m$. These variables must be specified on each isentropic surface on a ($\lambda$, $q$) grid. Friction, diabatic heating, and then $\dot{q}$ must be parameterized. Technical difficulties arise in diagnosing the Montgomery streamfunction and the pressure, by integrating (17) and (16), respectively, since their calculations involve double integrals over the horizontal and vertical domain.

Additional challenges are related to the boundary conditions. Along the earth's surface, the boundary conditions on PV are influenced by the variations in potential temperature (e.g., Bretherton 1966; Schneider et al. 2003; Schneider 2005), so it is difficult for PV to serve as a coordinate there. Therefore, at the surface a modified version of PV is necessary in order to smooth the singularities. The definition of surface PV proposed by Schneider et al. (2003) is adopted, which under no-slip boundary conditions reduces to

$$q_s = \frac{2\Omega\mu}{\sigma}\theta\delta(\theta - \theta_s), \quad (22)$$

where subscript $s$ refers to the surface.

Figure 2 shows a distribution of the $\mu$ contours as a function of $q$ and $\theta$. This distribution was obtained from the climatological mean of the zonally averaged (at fixed latitude) PV expressed on isentropic surfaces. The data used are monthly mean values of the National Centers for Environmental Prediction–National Center for Atmospheric Research (NCEP–NCAR) Reanalysis (Kalnay et al. 1996). In this figure we identify two problems: one is the fluctuating PV position of the geographical poles and the other is the shape of the domain spanned by PV when it is used as a coordinate. The former problem leads to the idea to develop a hybrid meridional coordinate that behaves as latitude near the poles and has the characteristics of PV coordinates near the equator. This issue will be the subject of a subsequent paper.

### c. *Barotropic adjustment*

One of the features of a model that uses PV as meridional coordinate is the built-in barotropic adjustment that prevents the folding of PV surfaces. The adjustment mechanism is similar to the built-in dry convective adjustment that occurs in an isentropic model (e.g., Hsu and Arakawa 1990; Konor and Arakawa 1997).

Eliassen (1983) extended the Charney–Stern theorem (Charney and Stern 1962) and showed that a necessary condition for barotropic instability is that the meridional gradient of PV does not have the same sign everywhere on an isentropic surface. In a model that uses latitude as the meridional coordinate and predicts the PV, the onset of barotropic instability can be considered to occur when the meridional component of the PV gradient changes its sign. This is illustrated in Fig. 3, which shows the result of a very simple calculation, based on (9), in which PV on an isentropic surface changes only in response to a prescribed $\dot{q}$; the horizontal and vertical advection are assumed zero. The model is one-dimensional and assumes a periodic domain. Potential vorticity is predicted at each grid point and the PV gradient, $(\partial q/\partial\mu)$, is diagnosed at each time step. The latitudinal profile of $\dot{q}$ is shown in Fig. 4 and its form is designed to induce $(\partial q/\partial\mu) < 0$ when it is applied to a balanced PV field. The evolution in time of the PV gradient shows a barotropically stable state that

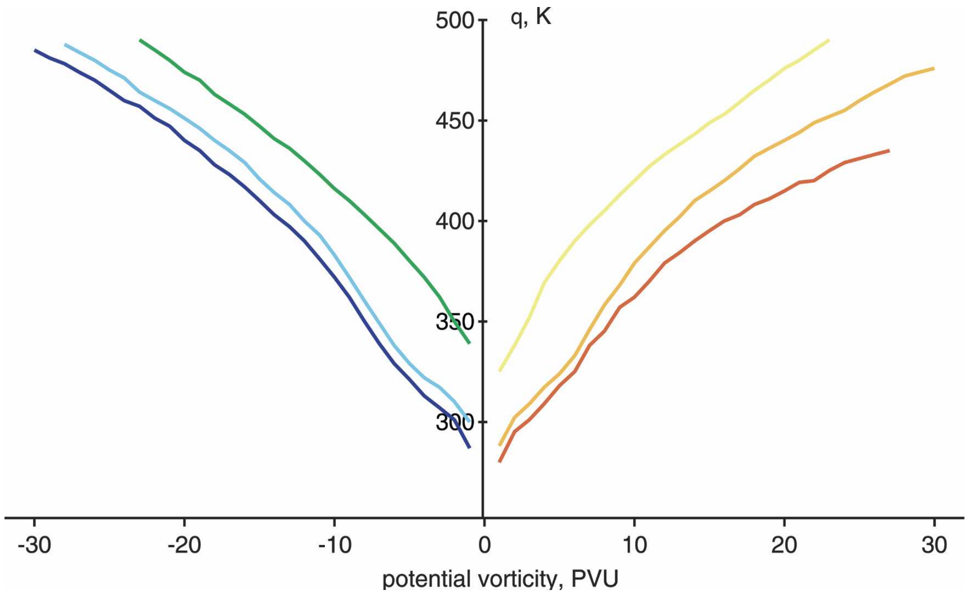


FIG. 2. Typical distribution of $\mu$ on the potential vorticity, potential temperature cross section derived from climatological zonal mean values of potential vorticity NCEP–NCAR reanalysis. The dark blue line correspond to $\mu = -1$, light blue to $\mu = -\sqrt{3}/2$, green to $\mu = -1/2$, yellow to $\mu = 1/2$, orange to $\mu = \sqrt{3}/2$, and red to $\mu = 1$.

tends to become unstable in response to the applied forcing $\dot{q}$.

Figure 5 shows the results of a model that is based on a simplified version of the continuity equation

$$\left(\frac{\partial h}{\partial t}\right)_{\lambda,q,\theta} = -\frac{1}{a}\left\{\frac{\partial}{\partial q}(h\dot{q})\right\}_{\lambda,\theta}. \tag{23}$$

In (23), besides neglecting the zonal and vertical advection, $\sigma$ is assumed constant. The evolution in time of the PV thickness shows that, as the model approaches a barotropic-unstable state, $h$ becomes very large, eventually diverging to infinity as barotropic instability sets in. Therefore, in order to go from barotropic stability to barotropic instability, the model has to make the PV

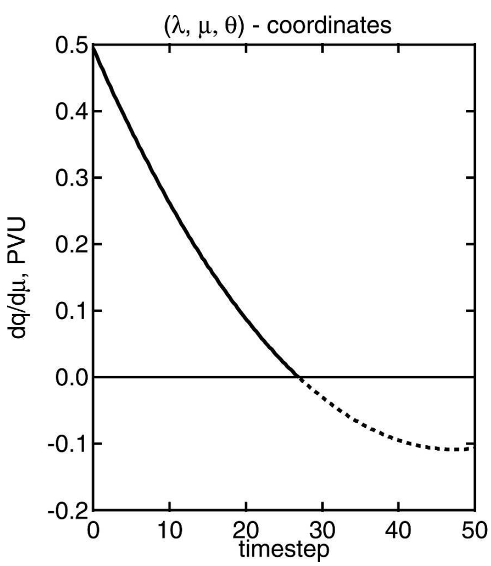


FIG. 3. Evolution in time of the meridional potential vorticity gradient ($dq/d\mu$) in a model that uses latitude as meridional coordinate. The onset of barotropic instability is considered the time step when the gradient becomes negative.

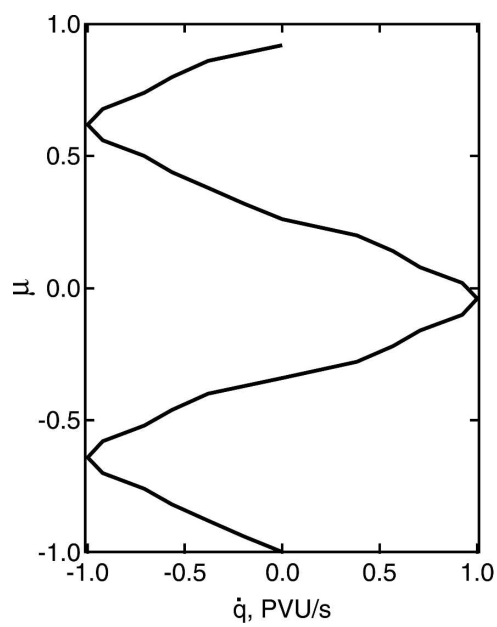


FIG. 4. The idealized meridional distribution of prescribed $\dot{q}$.

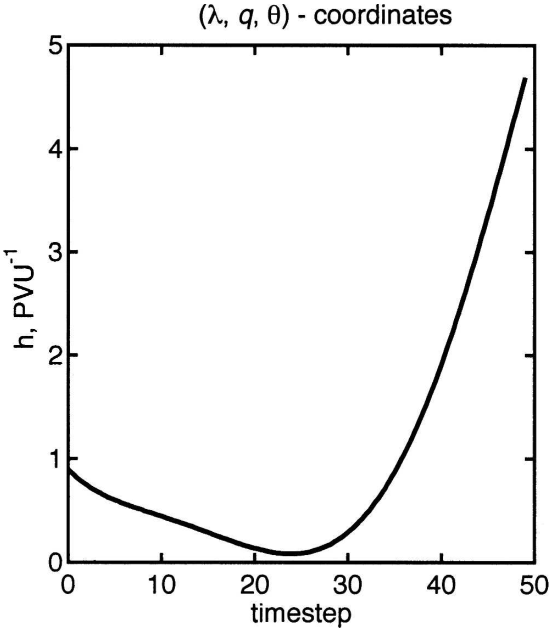


FIG. 5. Evolution in time of the PV thickness ($h$) in a model that uses potential vorticity as meridional coordinate. As the model approaches barotropic instability the PV thickness becomes very large.

thickness to go to plus infinity and then come back from minus infinity. In the barotropic adjustment mechanism the air parcels are arranged in such a way that the PV thickness does not become negative. This is what we mean when we say that the barotropic adjustment is a built-in process.

The atmospheric counterpart of the barotropic adjustment is the mixing of potential vorticity by baroclinic eddies that are triggered by the baroclinic instability. Both observational and numerical studies (e.g., Sun and Lindzen 1994; Stone and Nemet 1996) show that the eddies mix potential vorticity. The baroclinic–barotropic adjustment has been described by Stone (1978) as a process in which baroclinic eddies destroy themselves in the interaction with the mean flow by adjusting the mean temperature field and altering the eddy available potential energy necessary for further growth.

## 3. A shallow-water model using PV as meridional coordinate

A shallow-water model with a free surface can be formulated in which the role of latitude is taken by the PV, defined as

$$q = \frac{1}{H}\left\{2\Omega\mu + \frac{1}{a(1-\mu^2)}\left(\frac{\partial V}{\partial \lambda}\right)_\mu - \frac{1}{a}\left(\frac{\partial U}{\partial \mu}\right)_\lambda\right\}, \tag{24}$$

where $H$ is the free surface of the fluid in the absence of topography and all the other notations are the same as in section 2.

Using the rules of transformations descried by (11)–(13) the governing equations describing the shallow-water fluid in the absence of friction is

$$\frac{DU}{Dt} - 2\Omega\mu V + \frac{g}{a}\left\{\left(\frac{\partial H}{\partial \lambda}\right)_q - \frac{1}{h}\left(\frac{\partial H}{\partial q}\right)_\lambda\left(\frac{\partial \mu}{\partial \lambda}\right)_q\right\} = 0, \tag{25}$$

$$\frac{DV}{Dt} + 2\mu\left(\Omega U + \frac{K}{a}\right) + \frac{1-\mu^2}{ah}\left(\frac{\partial H}{\partial q}\right)_\lambda = 0, \tag{26}$$

$$\left(\frac{\partial hH}{\partial t}\right)_{\lambda,q} + \frac{1}{a}\left\{\frac{\partial}{\partial \lambda}\left(\frac{hHU}{1-\mu^2}\right)\right\}_q = 0, \tag{27}$$

$$\frac{D\mu}{Dt} - \frac{V}{a} = 0, \tag{28}$$

where the Lagrangian derivative has a very simple form because in the absence of friction PV is materially conserved

$$\frac{D}{Dt} = \left(\frac{\partial}{\partial t}\right)_{\lambda,q} + \frac{U}{a(1-\mu^2)}\left(\frac{\partial}{\partial \lambda}\right)_q. \tag{29}$$

For a nondivergent barotropic model, the continuity equation (27) reduces to the same form as Kushner and Held (1999) derived on a plane.

### *a. The equatorial $\beta^*$-plane approximation*

The $\beta^*$-plane approximation differs from what is usually called the $\beta$-plane approximation through the way in which the Coriolis parameter is expanded in Taylor series. In the PV coordinate $f \approx f_0 + \beta^* q$, where $\beta^* = (df/dq)_{y_0}$, and $q = 0$ at the equator.

To derive the $\beta^*$-plane approximation, a geometrical simplification is adopted in which the spherical coordinates ($\lambda$, $q$, $t$) are replaced with the Cartesian coordinates ($x$, $q$, $t$). In this frame of reference $a\ \mu = y$. Since this problem is often studied in a nondimensional form (Matsuno 1966), $c = (g\overline{H})^{1/2}$ is adopted as the constant gravity wave speed based on the mean depth $\overline{H}$. The horizontal length scales are defined as $L = (c\overline{h}/\beta^*)^{1/2}$ and $L_q = (c/\overline{h}\beta^*)^{1/2}$, and the unit of time is $T = (\overline{h}/c\beta^*)^{1/2}$. The nondimensional version of (25)–(28) linearized about a basic state of rest in a domain close to the equator is

$$\frac{\partial u}{\partial t} - qv + \left(\frac{\partial H}{\partial x}\right)_q = 0, \qquad (30)$$

$$\frac{\partial v}{\partial t} + qu + \left(\frac{\partial H}{\partial q}\right)_x = 0, \qquad (31)$$

$$\frac{\partial H}{\partial t} + \frac{\partial v}{\partial q} + \left(\frac{\partial u}{\partial x}\right)_q = 0, \qquad (32)$$

where $(u, v)$ are the dimensionless perturbations of the wind, $H$ is now the nondimensional perturbation of the free surface of the shallow-water fluid, and $h$ is the perturbation of the PV thickness and is also nondimensional. Assuming wavelike solutions of the form $A(x, q, t) = \mathcal{A}(k, q)\, e^{i(kx+\nu t)}$, where $k$ is the zonal wavenumber and $\nu$ is the frequency, the system of Eqs. (30)–(32) reduces to

$$\frac{d^2\mathcal{V}(k, q)}{dq^2} + \left(\nu^2 - k^2 + \frac{k}{\nu} - q^2\right)\mathcal{V}(k, q) = 0. \qquad (33)$$

This is similar to the equation describing the quasigeostrophic motion on the $\beta$ plane obtained by Matsuno (1966). If $v = 0$, (30)–(32) give the solution corresponding to Kelvin waves. This demonstrates that a nondivergent barotropic model using potential vorticity as meridional coordinate is able to simulate inertio-gravity, Rossby, and Kelvin waves, which dominate in the equatorial area. It also illustrates that use of the PV coordinate does not prevent the model from representing unbalanced motions, such as, gravity waves and Kelvin waves.

### *b. Zonally averaged model*

The zonal average along a potential vorticity contour for an arbitrary function $A(\lambda, q, t)$ is defined as

$$[A(q, t)] = \frac{1}{2\pi}\int_0^{2\pi} A(\lambda, q, t)\, d\lambda, \qquad (34)$$

and the deviation from the zonal mean is

$$A^* = A - [A]. \qquad (35)$$

To get new insight into the physical mechanisms affecting the zonal mean flow in the shallow-water model, it is useful to express the governing equations in flux form before taking the zonal average. It can easily be shown that the zonally averaged equations describing the zonal mean flow in a shallow-water model that uses potential vorticity as meridional coordinate are

$$\frac{\partial}{\partial t}[hHL] = \frac{g}{a}\frac{\partial}{\partial q}\left[\frac{(H^2)^*}{2}\left(\frac{\partial \mu^*}{\partial \lambda}\right)_q\right], \qquad (36)$$

$$\frac{\partial}{\partial t}[hH] = 0, \qquad (37)$$

where $L$ represents the angular momentum per unit mass and is given by

$$L = a\{U + \Omega(1 - \mu^2)a\}. \qquad (38)$$

The continuity equation has a very simple form and shows that the mass-weighted zonally averaged free surface height remains at its initial value along each potential vorticity contour. The angular momentum equation (36) reveals a novelty that comes with the use of the potential vorticity coordinate. The zonally averaged mass-weighted angular momentum changes only in response to the net form drag acting on the slanted potential vorticity contour.

Figure 6 shows the distributions of the Montgomery streamfunction, deviation about the zonal mean, and PV contours on the isentropic surface 315 K for 1 January 2005. The data used are daily values of the NCEP–NCAR reanalysis (Kalnay et al. 1996). In the absence of diabatic heating an isentropic layer is analogous to a shallow-water model. One can notice that high pressure regions lie east and low pressure regions lie west of the high PV air. According to this the eastern side of a surface of constant PV has higher pressure than the western side. Hence, a net westward form drag is exerted on the PV surface, as schematically illustrated in Fig. 7. This result has been applied to shed some light on the physical mechanisms that maintain the mean meridional circulation in a stratified fluid as described in section 2b. In the next section, we derive a similar result for the full PVPT system.

## 4. Interactions between eddies and the mean flow, as seen in PVPT coordinates

To investigate the nature of interactions between the eddies and the mean flow, we consider the zonal average of the angular momentum principle expressed in flux form, which is given by

$$\frac{\partial [mL]}{\partial t} + \frac{1}{a}\frac{\partial}{\partial q}[\dot{q}mL] + \frac{\partial}{\partial \theta}[\dot{\theta}mL]$$
$$= [mX_\lambda] + \frac{1}{a}\frac{\partial}{\partial q}\left[F^*\left(\frac{\partial \mu^*}{\partial \lambda}\right)_{q,\theta} - G\left(\frac{\partial \mu}{\partial \theta}\right)_{\lambda,q}\right]$$
$$+ \frac{1}{a}\frac{\partial}{\partial \theta}\left[G\left(\frac{\partial \mu}{\partial q}\right)_{\lambda,\theta}\right], \qquad (39)$$

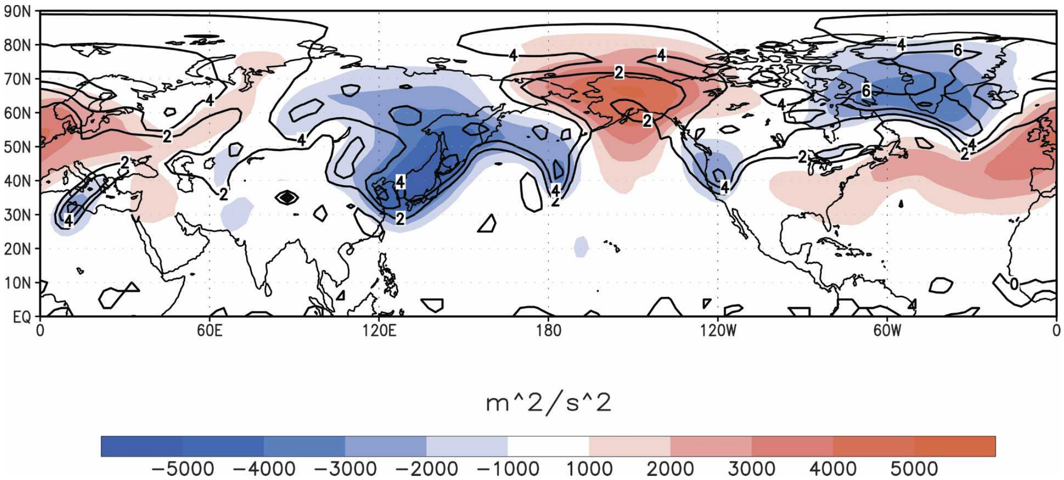


FIG. 6. Northern Hemisphere distributions of the Montgomery streamfunction (shaded regions), deviation about the zonal mean, and contours of potential vorticity [contour interval = 2 PVU, where 1 PV unit (PVU) = 1.08 × $10^{-6}$ m$^2$ s$^{-1}$ K kg$^{-1}$] on the isentropic surface $\theta$ = 315 K for 1 Jan 2005 in NCEP–NCAR reanalysis data.

where $F$ is a function of pressure only and satisfies the identities

$$\frac{p}{g}\frac{\partial \Pi}{\partial \lambda} \equiv \frac{\partial F}{\partial \lambda}, \quad \frac{p}{g}\frac{\partial \Pi}{\partial q} \equiv \frac{\partial F}{\partial q}, \quad \frac{p}{g}\frac{\partial \Pi}{\partial \theta} \equiv \frac{\partial F}{\partial \theta}. \tag{40}$$

The function $G$ is defined by

$$G(\lambda, q, \theta) \equiv \theta\left\{\left(\frac{\partial F}{\partial \lambda}\right)_{q,\theta} - \frac{1}{h}\left(\frac{\partial F}{\partial q}\right)_{\lambda,\theta}\left(\frac{\partial \mu}{\partial \lambda}\right)_{q,\theta}\right\} + p\left\{\left(\frac{\partial z}{\partial \lambda}\right)_{q,\theta} - \frac{1}{h}\left(\frac{\partial z}{\partial q}\right)_{\lambda,\theta}\left(\frac{\partial \mu}{\partial \lambda}\right)_{q,\theta}\right\}, \tag{41}$$

and can be also written as

$$G(\lambda, q, \theta) = \frac{p}{g}\left(\frac{\partial M}{\partial \lambda}\right)_{\mu,\theta}. \tag{42}$$

A full derivation of the right-hand side of (39) is given in the appendix.

Equation (39) shows that, in the absence of diabatic heating and friction, zonally averaged mass-weighted angular momentum changes only in response to the form drag acting against the walls of the PVPT tubes. The physical interpretation as form drag of $G(\partial\mu/\partial\theta)_{\lambda,q}$ and $G(\partial\mu/\partial q)$ becomes evident when, using (42), we rewrite the two terms as

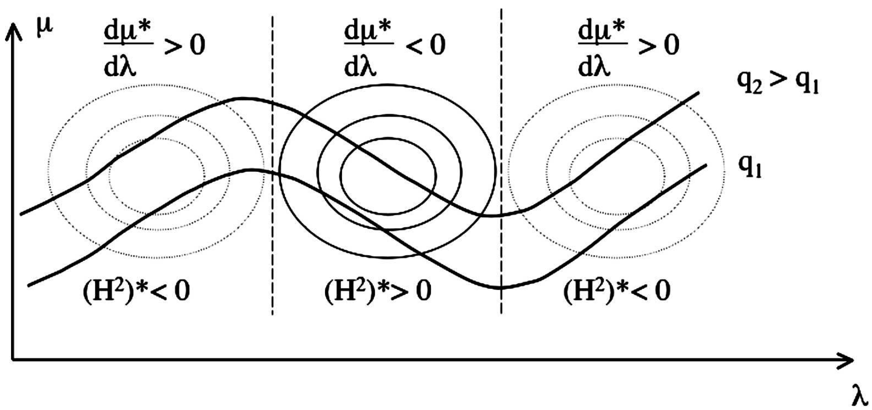


FIG. 7. Schematic representation of the form drag acting on the air that flows through a channel bounded by surfaces of constant PV, $q_1$, and $q_2$. Contour lines of $(H^2)^*$ denotes deviation of the height of the fluid from the zonal mean height, whereas $\partial\mu^*/\partial\lambda$ gives the slope of the PV contour, and they are negatively correlated everywhere. As a consequence, in a zonally averaged sense as the air moves toward the east it is pushed back by the pressure force exerted by the undulating PV contours.

$$G\left(\frac{\partial\mu}{\partial\theta}\right)_{\lambda,q} = \left[\frac{p}{g}\left(\frac{\partial\mu}{\partial\theta}\right)_{\lambda,q}\left(\frac{\partial M}{\partial\lambda}\right)_{\mu,\theta}\right] = \left[p^*_\theta\left(\frac{\partial M^*}{\partial\lambda}\right)_{\mu,\theta}\right]$$
$$G\left(\frac{\partial\mu}{\partial q}\right)_{\lambda,\theta} = \left[\frac{p}{g}\left(\frac{\partial\mu}{\partial q}\right)_{\lambda,\theta}\left(\frac{\partial M}{\partial\lambda}\right)_{\mu,\theta}\right] = \left[p^*_q\left(\frac{\partial M^*}{\partial\lambda}\right)_{\mu,\theta}\right], \qquad (43)$$

where $p_\theta$ and $p_q$ represent the components of the modified pressure force that acts on the PV contours. If we assume that $G = G(\lambda)$, the zonally averaged angular momentum changes only in response to the meridional variation of the form drag, and, when $F$ is a zonally symmetric function, the angular momentum is conserved in the absence of diabatic heating and friction.

In steady-state conditions and in the absence of diabatic heating and friction, the left-hand side of (39) is zero so that the equation reduces to

$$\nabla \cdot \mathbf{E} = 0$$

where $\mathbf{E} = (0, E^q, E^\theta)$ is the Eliassen–Palm flux vector, with the meridional and vertical components given by

$$E^q = \left[F^*\left(\frac{\partial\mu^*}{\partial\lambda}\right)_{q,\theta}\right] - \left[G\left(\frac{\partial\mu}{\partial\theta}\right)_{\lambda,q}\right],$$
$$E^\theta = \left[G\left(\frac{\partial\mu}{\partial q}\right)_{\lambda,\theta}\right]. \qquad (44)$$

Following Andrews (1983) we interpret $\nabla \cdot \mathbf{E}$ as the zonal component of the forces exerted by the eddies on a thin tube bounded by undulating lateral sides, which are located at $q$ and $q + dq$, and undulating bottom and top isentropes $\theta$ and $\theta + \mathrm{d}\theta$. The form drag nature of the vertical component of the Eliassen–Palm flux was also pointed out by Andrews (1983) using isentropic coordinates, and by Iwasaki (2001) using the isentropic zonal mean pressure as a vertical coordinate.

The zonally averaged continuity equation is

$$\frac{\partial[m]}{\partial t} + \frac{1}{a}\frac{\partial}{\partial q}[m\dot{q}] + \frac{\partial}{\partial\theta}[m\dot{\theta}] = 0. \qquad (45)$$

This shows that mass contained in a zonally averaged PVPT tube can be changed only by diabatic and frictional processes. In the absence of these two processes the zonally averaged mass is constant along each contour on an isentropic surface. When diabatic processes and friction are present, (45) can be rewritten as

$$\frac{\partial[m]}{\partial t} + \frac{1}{a}\frac{\partial}{\partial q}\hat{q}[m] + \frac{\partial}{\partial\theta}\hat{\dot{\theta}}[m] = 0, \qquad (46)$$

where $\hat{q}$ and $\hat{\dot{\theta}}$ are mass-weighted zonally averaged quantities given by

$$\hat{q} = \frac{[m\dot{q}]}{[m]}, \quad \hat{\dot{\theta}} = \frac{[m\dot{\theta}]}{[m]}. \qquad (47)$$

In steady-state conditions, $(\hat{q}, \hat{\dot{\theta}})$ represent a residual circulation that tends to maintain the balanced state that would be otherwise disturbed by the action of eddies. The vertical branches of the residual circulation are forced by differential heating, whereas the meridional branches are the response to heating and friction. Since there are no fluctuations of $q$ or $\theta$ along surfaces of constant PV and potential temperature, there are no eddy fluxes of these quantities in PVPT coordinates. The eddies affect the circulation only through form drag, as discussed above.

## 5. Conclusions and discussion

The present study provides a new perspective of the general circulation of the atmosphere using potential vorticity as meridional coordinate and potential temperature as vertical coordinate. In this framework, the meridional and vertical advections are zero under frictionless adiabatic processes. Thus, the air flows zonally along a contour of constant potential vorticity on an isentropic surface. In the PVPT system of coordinates, the atmosphere is divided into undulating tubes that are bounded on the top and bottom by isentropic surfaces and on the sides by surfaces of constant potential vorticity. In the Tropics, PVPT tubes are deep and wide because the static stability is small in this region and PV varies slowly. In middle latitudes the PVPT tubes are shallow and narrow because meridional PV gradients are large and potential temperature varies fast with height.

The PVPT frame of reference allows a Lagrangian description of the flow similar to the generalized Lagrangian mean (GLM) formalism proposed by McIntyre (1980). In the GLM theory, the averaging is applied along the trajectory followed by the particle. The PV coordinate tries to follow the trajectory of the particle because PV is materially conserved in the absence of diabatic and factional processes.

The meridional component of the wind, $V$, not to be confused with the projection of the wind vector on the PVPT coordinates, represents the velocity of a fluid parcel that conserves its potential vorticity moving on an isentropic surface. The primitive equations in the PVPT coordinates consist of four independent prognostic equations, rather than three as in the isentropic system. The additional prognostic equation provides information on the geographical latitude of a constant potential vorticity contour. This equation should not be interpreted as containing information about the coordinate trajectory. It is important to stress that we have not assumed a quasigeostrophic balance. However, monotonic variations of PV with latitude and potential temperature with height are assumed.

A model that uses the PVPT system of coordinates incorporates built-in dry convective and barotropic adjustment processes, which prevent the model from explicitly simulating dry-convective and barotropic–baroclinic instability. Therefore the atmospheric flow simulated by such a model can capture the characteristics of the flow before and after barotropic–baroclinic instability occurs but cannot be used to study the life cycle of baroclinic waves. We are exploring a generalized coordinate that allows barotropic–baroclinic instability to take place although the meridional coordinates remains a monotonic function of latitude.

The linearized equations of the shallow-water model on the equatorial beta plane contain the dominant equatorial disturbances; that is, Rossby, Kelvin, and inertio-gravity waves. The zonally averaged shallow-water model consists of two simple equations, which show that the mass weighted angular momentum changes only in response to the net form drag acting on a potential vorticity contour, and in a steady state the form drag is the same on all contours.

The PVPT frame of reference shows that the interaction between the eddy momentum flux and the zonal mean flow results in a deceleration of the latter due to the form drag exerted by the former on the walls of the PVPT tubes. Thus, in the PVPT tubes the eddies have a similar effect on the flow as a mountain has when the mean meridional circulation is studied in pressure or height coordinates. In PVPT coordinates, the zonal mean flow is driven by the form drag on the air that flows through the PVPT tubes, and the eddies modulate the shapes of the tubes. PVPT coordinates do not simplify the zonally averaged dynamical equations, but they provide a basis for assessing the physical importance of various terms in these equations.

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## APPENDIX

### Derivation of the Horizontal Pressure Gradient Term in the PVPT System of Coordinates

The flux form of horizontal pressure gradient term in PVPT coordinates is

$$\mathrm{HPGF} = \sigma h\left(\frac{\partial M}{\partial \lambda}\right)_{q,\theta} - \sigma\left(\frac{\partial M}{\partial q}\right)_{\lambda,\theta}\left(\frac{\partial \mu}{\partial \lambda}\right)_{q,\theta}. \tag{A1}$$

Using the definitions of pseudodensity and Montgomery streamfunction the first term can be written as

$$\sigma h\left(\frac{\partial M}{\partial \lambda}\right)_{q,\theta} = \left[-h\left(\frac{\partial p}{\partial \theta}\right)_{\lambda,q} + \left(\frac{\partial p}{\partial q}\right)_{\lambda,\theta}\left(\frac{\partial \mu}{\partial \theta}\right)_{\lambda,q}\right] \times \left[\frac{\theta}{g}\left(\frac{\partial \Pi}{\partial \lambda}\right)_{q,\theta} + \left(\frac{\partial z}{\partial \lambda}\right)_{q,\theta}\right]. \tag{A2}$$

Invoking the hydrostatic balance (18) and after few algebraic manipulations (A2) becomes

$$\begin{aligned}\sigma h\left(\frac{\partial M}{\partial \lambda}\right)_{q,\theta} = &-\frac{\partial}{\partial \theta}\left[\theta h\frac{p}{g}\left(\frac{\partial \Pi}{\partial \lambda}\right)_{q,\theta} + ph\left(\frac{\partial z}{\partial \lambda}\right)_{q,\theta}\right] + \frac{\partial}{\partial q}\left[\theta\frac{p}{g}\left(\frac{\partial \Pi}{\partial \lambda}\right)_{q,\theta}\left(\frac{\partial \mu}{\partial \theta}\right)_{\lambda,q} + p\left(\frac{\partial \mu}{\partial \theta}\right)_{\lambda,q}\left(\frac{\partial z}{\partial \lambda}\right)_{q,\theta}\right]\\ &-\frac{p}{h}\left[\frac{\theta}{g}\left(\frac{\partial \Pi}{\partial q}\right)_{\lambda,\theta} + \left(\frac{\partial z}{\partial q}\right)_{\lambda,\theta}\right]\left(\frac{\partial h}{\partial \lambda}\right)_{q,\theta}\left(\frac{\partial \mu}{\partial \theta}\right)_{\lambda,q} + p\left[\frac{\theta}{g}\left(\frac{\partial \Pi}{\partial q}\right)_{\lambda,\theta} + \left(\frac{\partial z}{\partial q}\right)_{\lambda,\theta}\right]\frac{\partial^2 \mu}{\partial \lambda \partial \theta} + h\frac{p}{g}\left(\frac{\partial \Pi}{\partial \lambda}\right)_{q,\theta}.\end{aligned} \tag{A3}$$

Applying the same recipe onto the second term of (A1) we obtain

$$\begin{aligned}\sigma\left(\frac{\partial M}{\partial q}\right)_{\lambda,\theta}\left(\frac{\partial \mu}{\partial \lambda}\right)_{q,\theta} = &-\frac{\partial}{\partial \theta}\left[\theta\frac{p}{g}\left(\frac{\partial \Pi}{\partial q}\right)_{\lambda,\theta}\left(\frac{\partial \mu}{\partial \lambda}\right)_{q,\theta} + p\left(\frac{\partial z}{\partial q}\right)_{\lambda,\theta}\left(\frac{\partial \mu}{\partial \lambda}\right)_{q,\theta}\right]\\ &+\frac{\partial}{\partial q}\left[\frac{p}{g}\left(\frac{\partial \Pi}{\partial q}\right)_{\lambda,\theta}\frac{\theta}{h}\left(\frac{\partial \mu}{\partial \theta}\right)_{\lambda,q}\left(\frac{\partial \mu}{\partial \lambda}\right)_{q,\theta} + \frac{p}{h}\left(\frac{\partial z}{\partial q}\right)_{\lambda,\theta}\left(\frac{\partial \mu}{\partial \theta}\right)_{\lambda,q}\left(\frac{\partial \mu}{\partial \lambda}\right)_{q,\theta}\right]\\ &-\frac{p}{h}\left[\frac{\theta}{g}\left(\frac{\partial \Pi}{\partial q}\right)_{\lambda,\theta} + \left(\frac{\partial z}{\partial q}\right)_{\lambda,\theta}\right]\left(\frac{\partial h}{\partial \lambda}\right)_{q,\theta}\left(\frac{\partial \mu}{\partial \theta}\right)_{\lambda,q} + p\left[\frac{\theta}{g}\left(\frac{\partial \Pi}{\partial q}\right)_{\lambda,\theta} + \left(\frac{\partial z}{\partial q}\right)_{\lambda,\theta}\right]\frac{\partial^2 \mu}{\partial \lambda \partial \theta}\\ &+\frac{p}{g}\left(\frac{\partial \Pi}{\partial q}\right)_{\lambda,\theta}\left(\frac{\partial \mu}{\partial \lambda}\right)_{q,\theta}.\end{aligned} \tag{A4}$$

Combining (A3) and (A4) the horizontal pressure gradient term becomes

$$\begin{aligned}\mathrm{HPGF} = &-\frac{\partial}{\partial\theta}\left[\theta h\frac{p}{g}\left(\frac{\partial\Pi}{\partial\lambda}\right)_{q,\theta} + ph\left(\frac{\partial z}{\partial\lambda}\right)_{q,\theta} - \theta\frac{p}{g}\left(\frac{\partial\Pi}{\partial q}\right)_{\lambda,\theta}\left(\frac{\partial\mu}{\partial\lambda}\right)_{q,\theta} - p\left(\frac{\partial z}{\partial q}\right)_{\lambda,\theta}\left(\frac{\partial\mu}{\partial\lambda}\right)_{q,\theta}\right]\\ &+\frac{\partial}{\partial q}\left[\theta\frac{p}{g}\left(\frac{\partial\Pi}{\partial\lambda}\right)_{q,\theta}\left(\frac{\partial\mu}{\partial\theta}\right)_{\lambda,q} + p\left(\frac{\partial\mu}{\partial\theta}\right)_{\lambda,q}\left(\frac{\partial z}{\partial\lambda}\right)_{q,\theta} - \frac{p}{g}\left(\frac{\partial\Pi}{\partial q}\right)_{\lambda,\theta}\frac{\theta}{h}\left(\frac{\partial\mu}{\partial\theta}\right)_{\lambda,q}\left(\frac{\partial\mu}{\partial\lambda}\right)_{q,\theta}\right.\\ &\left.-\frac{p}{h}\left(\frac{\partial z}{\partial q}\right)_{\lambda,\theta}\left(\frac{\partial\mu}{\partial\theta}\right)_{\lambda,q}\left(\frac{\partial\mu}{\partial\lambda}\right)_{q,\theta}\right] + h\frac{p}{g}\left(\frac{\partial\Pi}{\partial\lambda}\right)_{q,\theta} - \frac{p}{g}\left(\frac{\partial\Pi}{\partial q}\right)_{\lambda,\theta}\left(\frac{\partial\mu}{\partial\lambda}\right)_{q,\theta}.\end{aligned} \tag{A5}$$

Defining $F(p)$ such that (40) is satisfied, after few steps (A5) becomes

$$\begin{aligned}\mathrm{HPGF} = &-\frac{\partial}{\partial\theta}\left\{\theta h\left[\left(\frac{\partial F}{\partial\lambda}\right)_{q,\theta} - \frac{1}{h}\left(\frac{\partial F}{\partial q}\right)_{\lambda,\theta}\left(\frac{\partial\mu}{\partial\lambda}\right)_{q,\theta}\right] + ph\left[\left(\frac{\partial z}{\partial\lambda}\right)_{q,\theta} - \frac{1}{h}\left(\frac{\partial z}{\partial q}\right)_{\lambda,\theta}\left(\frac{\partial\mu}{\partial\lambda}\right)_{q,\theta}\right]\right\}\\ &+\frac{\partial}{\partial q}\left\{-F\left(\frac{\partial\mu}{\partial\lambda}\right)_{q,\theta} + \theta\left(\frac{\partial\mu}{\partial\theta}\right)_{\lambda,q}\left[\left(\frac{\partial F}{\partial\lambda}\right)_{\lambda,\theta} - \frac{1}{h}\left(\frac{\partial F}{\partial q}\right)_{\lambda,\theta}\left(\frac{\partial\mu}{\partial\lambda}\right)_{q,\theta}\right]\right.\\ &\left.+p\left(\frac{\partial\mu}{\partial\theta}\right)_{\lambda,q}\left[\left(\frac{\partial z}{\partial\lambda}\right)_{q,\theta} - \frac{1}{h}\left(\frac{\partial z}{\partial q}\right)_{\lambda,\theta}\left(\frac{\partial\mu}{\partial\lambda}\right)_{q,\theta}\right]\right\} + \frac{\partial}{\partial\lambda}\left[F\left(\frac{\partial\mu}{\partial q}\right)_{\lambda,\theta}\right].\end{aligned} \tag{A6}$$

Introducing the notation (41), the horizontal pressure gradient force is

$$\begin{aligned}\mathrm{HPGF} = &\frac{\partial}{\partial\lambda}\left[F\left(\frac{\partial\mu}{\partial q}\right)_{\lambda,\theta}\right]\\ &+\frac{\partial}{\partial q}\left[-F\left(\frac{\partial\mu}{\partial\lambda}\right)_{q,\theta} + G\left(\frac{\partial\mu}{\partial\theta}\right)_{\lambda,q}\right]\\ &-\frac{\partial}{\partial\theta}\left[G\left(\frac{\partial\mu}{\partial q}\right)_{\lambda,\theta}\right].\end{aligned} \tag{A7}$$

To verify the correctness of the derived HPGF suppose the isolines of PV are perfectly lined up along latitudinal circles. As a consequence, (A7) reduces to

$$\begin{aligned}\mathrm{HPGF} = &-\theta\frac{\partial}{\partial\theta}\left[\left(\frac{\partial F}{\partial\lambda}\right)_{q,\theta}\left(\frac{\partial\mu}{\partial q}\right)_{\lambda,\theta}\right]\\ &-\frac{\partial}{\partial\theta}\left[p\left(\frac{\partial z}{\partial\lambda}\right)_{q,\theta}\left(\frac{\partial\mu}{\partial q}\right)_{\lambda,\theta}\right],\end{aligned} \tag{A8}$$

which for a one-to-one correspondence between $\mu$ and $q$ becomes

$$\mathrm{HPGF} = -\left(\frac{\partial\mu}{\partial q}\right)_{\lambda,\theta}\left\{\theta\frac{\partial}{\partial\theta}\left(\frac{\partial F}{\partial\lambda}\right)_{q,\theta} + \frac{\partial}{\partial\theta}\left[p\left(\frac{\partial z}{\partial\lambda}\right)_{q,\theta}\right]\right\}. \tag{A9}$$

The form of HPGF as given by (A9) corresponds with the HPGF in isentropic coordinates. In the absence of vertical variations,

$$\frac{\partial}{\partial\theta}\left[G\left(\frac{\partial\mu}{\partial q}\right)_{\lambda,\theta}\right] = 0, \quad \left(\frac{\partial\mu}{\partial\theta}\right)_{\lambda,q} = 0,$$

and the horizontal pressure gradient force reduces to the form corresponding to the shallow-water case.

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