

Virtual Temperature

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21.1 Definition

For dry air, the equation of state is

$$p_d = \rho_d R_d T \tag{21.1}$$

Similarly, water vapor obeys its own equation of state with its own gas constant:

$$e = \rho_v R_V T \tag{21.2}$$

The total pressure is

$$p = p_d + e \tag{21.3}$$

We see that

$$p = (\rho_d R_d + \rho_v R_v) T \tag{21.4}$$

$$\rho = \rho_d + \rho_v + \rho_l \tag{21.5}$$

We *define* the virtual temperature as satisfying the ideal gas law with the total pressure and the total density, and the gas constant for dry air:

$$\rho R_d T_v \equiv p\,. \tag{21.6}$$

Substituting on both sides of this equation, we obtain

$$\begin{aligned} (\rho_d + \rho_v + \rho_l) R_d T_v &= (\rho_d R_d + \rho_v R_v) T \\ &= \left(\rho_d + \frac{\rho_v}{\varepsilon}\right) R_d T \end{aligned} \tag{21.7}$$

where

$$\varepsilon = \frac{R_d}{R_v} \cong 0.622 . \tag{21.8}$$

This leads to

$$\begin{aligned} T_v &= T\left(\frac{\rho_d + \frac{\rho_v}{\varepsilon}}{\rho_d + \rho_v + \rho_l}\right) \\ &= T\left(\frac{1 + \frac{q}{\varepsilon}}{1 + q + l}\right) \end{aligned} \tag{21.9}$$

or

$$\begin{aligned} T_v &\cong T\left(1 + \frac{q}{\varepsilon}\right)(1 - q - l) \\ &\cong T\left(1 - q - l + \frac{q}{\varepsilon}\right) \\ &= T(1 + \delta q - l), \end{aligned} \tag{21.10}$$

where

$$\delta \equiv \frac{1 - \varepsilon}{\varepsilon} \cong 0.608 \ . \tag{21.11}$$

21.2 Buoyancy fluctuations on pressure surfaces

The virtual dry static energy is

$$s_v \equiv c_p T_v + gz . \tag{21.12}$$

We introduce the moist static energy

$$h \equiv c_p T + gz + Lq , \tag{21.13}$$

which is approximately conserved under both moist and dry adiabatic processes, even when precipitation is occuring. The total water mixing ratio, $q + l$, as also approximately conserved under both moist and dry adiabatic processes, although it is of course affected by precipitation.

Consider fluctuations at constant pressure, denoted by primes. We can write

$$h' = c_p T' + Lq' . \tag{21.14}$$

Here we neglect height fluctuations on the constant pressure surface. From (21.10) and (21.12), we see that

$$\begin{aligned} s_v' &\cong c_p T_v' \\ &\cong c_p T' + c_p \overline{T}[\delta q' - l'] \\ &= c_p T' + \varepsilon[\delta L q' - L l'] \end{aligned} \tag{21.15}$$

where

$$\varepsilon \equiv \frac{c_p \overline{T}}{L} . \tag{21.16}$$

This can be manipulated as follows:

$$\begin{aligned} s_v' &\cong (c_p T' + Lq') - (1 - \delta\varepsilon)Lq' - \varepsilon L l' \\ &= h' - (1 - \delta\varepsilon)L(q' + l') + [1 - (1 + \delta\varepsilon)]Ll' \end{aligned} \tag{21.17}$$

21.3 Buoyancy fluxes across presssure surfaces

Following Lilly (1968), we can construct an expression for the virtual dry static energy flux, F_{sv}, by multiplying (21.17) by $\rho w'$ and then averaging. (We neglect the fluctuations of ρ.) The result is

$$F_{sv} = F_h - (1 - \delta\varepsilon)LF_{q+l} + [1 - (1 + \delta\varepsilon)]LF_l . \tag{21.18}$$

Eq. (21.18) is valid regardless of the cloud amount.

Still following Lilly (1968), we consider two cases. First, if there is no cloud, then $LF_l = 0$ and (21.18) reduces to

$$F_{sv} = (F_{sv})_{\mathrm{clr}} \equiv F_h - (1 - \delta\varepsilon)LF_{q+l} . \tag{21.19}$$

If there is a uniform cloud, so that the air is saturated everywhere, we can write

$$\gamma c_p T' \cong Lq' , \tag{21.20}$$

where

$$\gamma \equiv \frac{L}{c_p}\left(\frac{\partial q_*}{\partial T}\right)_p . \tag{21.21}$$

From (21.20), we get

$$\gamma h' = (1+\gamma)Lq' , \tag{21.22}$$

or

$$Lq' = \left(\frac{\gamma}{1+\gamma}\right)h' . \tag{21.23}$$

It follows that

$$Ll' = L(q'+l') - \left(\frac{\gamma}{1+\gamma}\right)h' , \tag{21.24}$$

or

$$LF_l = LF_{q+l} - \left(\frac{\gamma}{1+\gamma}\right)F_h . \tag{21.25}$$

Substituting (21.25) into (21.18), we find that for the fully cloudy case

$$F_{sv} = (F_{sv})_{\text{cld}} \equiv \left[\frac{1+(1+\delta)\gamma\varepsilon}{1+\gamma}\right]F_h - \varepsilon LF_{q+l} . \tag{21.26}$$