The Shallow Water Equations

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1. A derivation of the shallow water equations

The shallow water equations are the simplest form of the equations of motion that can be used to describe the horizontal structure of an atmosphere. They describe the evolution of an incompressible fluid in response to gravitational and rotational accelerations. The solutions of the shallow water equations represent many types of motion, including Rossby waves and inertia-gravity waves. In these notes, we begin with a more complete version of the equations of motion, and simplify to obtain the shallow water equations.

Ignoring the effects of friction, the momentum equation is

$$\rho\left(\frac{D\mathbf{u}}{Dt} + f\mathbf{k}\times\mathbf{v}\right) = -\nabla p - \rho g\mathbf{k} \ . \tag{1}$$

Here ρ is the density of the air, $\mathbf{u}$ is the *three-dimensional* velocity vector, the total derivative operator $D(\ )/Dt$ is defined by

$$\frac{D(\ )}{Dt} \equiv \frac{\partial}{\partial t}(\ ) + (\mathbf{u}\cdot\nabla)(\ ) \ , \tag{2}$$

f is the coriolis parameter, $\mathbf{k}$ is a unit vector pointing away from the center of the planet, $\mathbf{v}$ is the horizontal velocity vector, $\mathbf{\Omega}$ is the angular velocity vector associated with the planet's rotation, p is pressure, and g is the magnitude of the acceleration vector due to the planet's gravity. In writing (1), we have adopted "traditional" approximations appropriate to an atmosphere whose depth is shallow compared with the radius of the planet.

The continuity equation is

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{u}) = 0 \ ; \tag{3}$$

it can also be written as

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{u} = 0 \ . \tag{4}$$

We now show how (1) and (4) can be simplified to obtain the shallow water equations.

It is convenient to partition pressure and density into equilibrium and departures from equilibrium values. That is,

$$p = p_0(z) + p'(x, y, z, t), \text{ and } \rho = \rho_0(z) + \rho'(x, y, z, t) \ . \tag{5}$$

The equilibrium pressure $p_0(z)$ and the equilibrium density $\rho_0(z)$ are defined in such a way that they satisfy the hydrostatic equation

$$\frac{d p_0}{dz} = -g \rho_0 \ . \tag{6}$$

Substituting (5) into the momentum equation (1), and using (6), we obtain the momentum equation in the form

$$\rho\left(\frac{D\mathbf{u}}{Dt} + f\mathbf{k} \times \mathbf{v}\right) = -\nabla p' - \rho' g\mathbf{k} \ . \tag{7}$$

The full density still appears on the left-hand side of (7); no approximations have been made in passing from (1) to (7). On the right-hand side of (7), we see a "buoyancy" term, which is the product of the perturbation density and the acceleration due to gravity.

Now we assume that the fluid is *incompressible*, that is, $\rho = \rho_0$ is a constant and $\rho' = 0$. Water is nearly incompressible, and the name "shallow water equations" comes partly from the use of this incompressibility assumption. For an incompressible fluid, the three-dimensional momentum equation, (7), can be simplified to

$$\rho\left(\frac{D\mathbf{u}}{Dt} + 2\,\mathbf{\Omega} \times \mathbf{v}\right) = -\nabla p' \ . \tag{8}$$

In addition, the mass conservation equation, (4), reduces to

$$\nabla \cdot \mathbf{u} = 0 \ . \tag{9}$$

Equations (8) and (9) apply at each point in the three-dimensional fluid.

Now we begin to consider how to describe the vertically integrated circulation. We assume that the effects of vertical shear of the horizontal velocity are negligible. Such effects would arise, for example, through the vertical advection of horizontal momentum. The "no-shear" assumption is reasonable if the fluid is "shallow," and this partly accounts for the name "shallow water equations."

The next step is to find the form of the horizontal pressure gradient force acting on the shallow, incompressible fluid. Supposed that there exists a "free" surface of constant or nearly constant pressure, which is also a material surface in the sense that no mass crosses it. This is analogous, for example, to the surface of the ocean. Let $h_f(x, y, t)$ be the height of a this free surface. Let $h_S(x, y)$ be the height of the surface topography and let $h^* = h_f - h_S$ be the depth of the fluid.

As mentioned above, we assume that no fluid crosses the free surface. It follows that the height of a parcel embedded in the free surface satisfies

$$\frac{Dh_f}{Dt} = w(x, y, h_f, t) \ . \tag{10}$$

Similarly, no fluid can cross the lower boundary, so the fluid motion there must follow the topography:

$$\frac{Dh_S}{Dt} = w_S \ , \tag{11}$$

or

$$\frac{\partial h_S}{\partial t} + \mathbf{v} \cdot \nabla h_S - w_S = 0 \ . \tag{12}$$

When the topographic height is independent of time, we obtain

$$w_S(x, y, h_S, t) = \mathbf{v} \cdot \nabla h_S \ . \tag{13}$$

Using our assumption that the density is constant, and integrating the hydrostatic equation from some arbitrary depth z within the fluid up to the free surface gives

$$p(x, y, z, t) - p(x, y, h_f, t) = -g\,\rho(h_f - z) \ . \tag{14}$$

As an upper boundary condition, we have

$$p(x, y, h_f, t) = p_f \ . \tag{15}$$

It follows that

$$p = g\,\rho(h_f - z) + p_f \ . \tag{16}$$

So, since $p = -g\,\rho z + p'(x,\,y,\,z,\,t)$, we obtain $p' = g\,\rho h_f + p_f$ and

$$\nabla p' = g\,\rho_0\,\nabla h_f \ . \tag{17}$$

Here we have assumed that p_f is horizontally uniform. Using (17), we can now write the horizontal momentum equation as

$$\frac{D\mathbf{v}}{Dt} + f\,\mathbf{k}\times\mathbf{v} = -g\,\nabla h_f \ . \tag{18}$$

Since we have assumed no vertical shear, this equation applies at any height within the fluid, but the range of heights in question is presumably small since the fluid is "shallow." Equation (18) is the form of the momentum equation for the shallow water system. Using the vector identity $(\mathbf{v}\cdot\nabla)\,\mathbf{v} = \nabla[(\mathbf{v}\cdot\mathbf{v})/2] + (\nabla\times\mathbf{v})\times\mathbf{v}$, we can rewrite the momentum equation (18) as

$$\frac{\partial\mathbf{v}}{\partial t}\,(\zeta + f)\,\mathbf{k}\times\mathbf{v} = \ \nabla\left(gh_f + K\right) \ . \tag{19}$$

Here

$$K \equiv \frac{\mathbf{v}\cdot\mathbf{v}}{2} \tag{20}$$

is the kinetic energy per unit mass, and

$$\zeta = \mathbf{k}\cdot(\nabla\times\mathbf{v}) \tag{21}$$

is the relative vorticity.

Since the horizontal velocity is independent of z, we can easily integrate the continuity equation, (9), with respect to z, from $z = h_S$ to $z = h_f$, to obtain

$$w(x,\,y,\,h_f,\,t) - w_S(x,\,y,\,h_S\,t) = -(h_f - h_S)\,\nabla\cdot\mathbf{v} \ . \tag{22}$$

where $\nabla\cdot\mathbf{v}$ is the horizontal divergence. Substitute (20) and (12) into (22), to obtain

$$\frac{\partial}{\partial t}\,(h_f - h_S) + \nabla\cdot[\mathbf{v}(h_f - h_S)] = 0 \ , \tag{23}$$

or, with $h^* = h_f - h_S$,

$$\frac{\partial}{\partial t} h^* + \nabla \cdot (\mathbf{v}h^*) = 0 \ . \tag{24}$$

This is the continuity equation for the shallow water system.

2. The vorticity and divergence equations for shallow water

The absolute vorticity is the sum of the relative and planetary vorticity, i.e.,

$$\eta \equiv \zeta + f \ . \tag{25}$$

The vorticity equation can be derived by applying the operator $\mathbf{k} \cdot \nabla \times$ to (19):

$$\begin{aligned}
\frac{\partial \zeta}{\partial t} &= \mathbf{k} \cdot \left[\nabla \times \frac{\partial \mathbf{v}}{\partial t} \right] \\
&= \mathbf{k} \cdot \nabla \times \left[-\eta \left(\mathbf{k} \times \mathbf{v} \right) - \nabla \left(g h_f + K \right) \right] \\
&= -\mathbf{k} \cdot \nabla \times \left[\eta \left(\mathbf{k} \times \mathbf{v} \right) \right] - \mathbf{k} \cdot \left[\nabla \times \nabla \left(g h_f + K \right) \right] \\
&= -\mathbf{k} \cdot \nabla \times [\eta (\mathbf{k} \times \mathbf{v})] \\
&= -\mathbf{k} \cdot \left\{ \nabla \eta \times \left(\mathbf{k} \times \mathbf{v} \right) + \eta \left[\nabla \times \left(\mathbf{k} \times \mathbf{v} \right) \right] \right\} \\
&= -\mathbf{k} \cdot \left[(\mathbf{v} \cdot \nabla \eta) - \mathbf{v} \left(\nabla \eta \cdot \mathbf{k} \right) \right] \\
&\quad -\mathbf{k} \cdot \left\{ \eta \, (\mathbf{v} \cdot \nabla) \, \mathbf{k} - \left(\mathbf{k} \cdot \nabla \right) \mathbf{v} + \mathbf{k}(\nabla \cdot \mathbf{v}) - \mathbf{v} \left(\nabla \cdot \mathbf{k} \right) \right\} \\
&= -(\mathbf{v} \cdot \nabla \eta) - \eta \, (\nabla \cdot \mathbf{v}) \\
&= -\nabla \cdot (\eta \, \mathbf{v}) \ .
\end{aligned} \tag{26}$$

Because the coriolis parameter does not depend on time, we can write $\partial \eta / \partial t = \partial \zeta / \partial t$. Then (26) can be re-written as

$$\frac{\partial \eta}{\partial t} = -\nabla \cdot (\eta \, \mathbf{v}) \ . \tag{27}$$

One of the conclusions that can be drawn from (27) is that if there is no absolute vorticity in the fluid at a certain time, then none will ever form. A purely divergent flow (in the absence of planetary rotation) will remain purely divergent forever, within the framework considered here.

The divergence is defined by

$$\delta \equiv \nabla \cdot \mathbf{v} \ . \tag{28}$$

The divergence equation can be derived from the momentum equation, as follows:

$$\begin{aligned}
\frac{\partial \delta}{\partial t} &= \nabla \cdot \frac{\partial v}{\partial t} \\
&= \nabla \cdot \left[-\eta \left(\mathbf{k} \times \mathbf{v} \right) - \nabla \left(g\, h_f + K \right) \right] \\
&= -\nabla \cdot \left[\eta \left(\mathbf{k} \times \mathbf{v} \right) - \nabla^2 \left(g h_f + K \right) \right] \\
&= -\left[\nabla \eta \cdot \left(\mathbf{k} \times \mathbf{v} \right) + \eta \nabla \cdot \left(\mathbf{k} \times \mathbf{v} \right) \right] - \nabla^2 \left(g h_f + K \right) \\
&= -\left\{ \nabla \eta \cdot \left(\mathbf{k} \times \mathbf{v} \right) + \eta \left[\mathbf{v} \cdot \left(\nabla \times \mathbf{k} \right) - \mathbf{k} \cdot (\nabla \times \mathbf{v}) \right] \right\} - \nabla^2 \left(g h_f + K \right) \\
&= \eta\, \mathbf{k} \cdot (\nabla \times \mathbf{v}) - \nabla \eta \cdot \left(\mathbf{k} \times \mathbf{v} \right) - \nabla^2 \left(g h_f + K \right) \\
&= \eta\, \mathbf{k} \cdot (\nabla \times \mathbf{v}) + \nabla \eta \cdot \left(\mathbf{v} \times \mathbf{k} \right) - \nabla^2 \left(g h_f + K \right) \\
&= \eta\, \mathbf{k} \cdot (\nabla \times \mathbf{v}) + \mathbf{k} \cdot (\nabla \eta \times \mathbf{v}) - \nabla^2 \left(g h_f + K \right) \\
&= \mathbf{k} \cdot [\nabla \eta \times \mathbf{v} + \eta\, (\nabla \times \mathbf{v})] - \nabla^2 \left(g h_f + K \right) \\
&= \mathbf{k} \cdot \nabla \times (\eta\, \mathbf{v}) - \nabla^2 \left(g h_f + K \right)
\end{aligned} \tag{29}$$

3. Expressing the shallow water equations in terms of the stream function and velocity potential

Helmholtz's Theorem states that any vector field $\mathbf{V}$ can be separated into rotational and divergent parts, i.e., $\mathbf{V} = \nabla\psi + \mathbf{V}_\chi$, where $\nabla \cdot \mathbf{V}_\psi = 0$ and $\nabla \times \mathbf{V}_\chi = 0$. If the vector field is the horizontal wind, we can define a stream function, ψ, to express the rotational part, $\mathbf{k}\times\nabla\psi$, and a velocity potential, χ, to express the divergent part, $\nabla\chi$, i.e.,

$$\mathbf{v} = \mathbf{k}\times\nabla\psi + \nabla\chi \ . \tag{30}$$

The conservation equation for absolute vorticity is can be rewritten in terms of the stream function and velocity potential, as follows:

$$\begin{aligned}\frac{\partial\eta}{\partial t} &= -\nabla\cdot(\eta\,\mathbf{v})\\ &= -\nabla\cdot\left[\eta\ \left(\mathbf{k}\times\nabla\psi + \nabla\chi\right)\right]\\ &= -\nabla\cdot\left[\eta\ \left(\mathbf{k}\times\nabla\psi\right)\right] - \nabla\cdot(\eta\ \nabla\chi)\\ &= -\nabla\cdot(\eta\ \nabla\chi) - \eta\ \nabla\cdot\left(\mathbf{k}\times\nabla\psi\right) - \nabla\eta\ \cdot\left(\mathbf{k}\times\nabla\psi\right)\\ &= -\nabla\cdot(\eta\ \nabla\chi) + \mathbf{k}\cdot(\nabla\eta\ \times\nabla\psi)\ .\end{aligned} \tag{31}$$

For arbitrary scalar functions A and B, define the Jacobian operator $J(A, B)$ by:

$$J(A, B) \equiv \mathbf{k}\cdot(\nabla A\times\nabla B)\ . \tag{32}$$

Then we can rewrite (31) as

$$\frac{\partial\eta}{\partial t} = -\nabla\cdot(\eta\,\nabla\chi) + J(\eta\,,\,\psi)\ . \tag{33}$$

The divergence equation can be re-written in terms of the stream function and velocity potential as follows:

$$\begin{aligned}\frac{\partial\delta}{\partial t} &= \mathbf{k}\ \cdot\nabla\times(\eta\,\mathbf{v}) - \nabla^2\left(gh_f + K\right)\\ &= \mathbf{k}\ \cdot\nabla\times\left[\eta\left(\mathbf{k}\times\nabla\psi + \nabla\chi\right)\right] - \nabla^2\left(gh_f + K\right)\end{aligned}$$

$$= \mathbf{k} \cdot \nabla \times \left[\eta \left(\mathbf{k} \times \nabla \psi \right) + \eta \nabla \chi \right] - \nabla^2 \left(g h_f + K \right)$$

$$= \mathbf{k} \cdot \nabla \times \left[\eta \left(\mathbf{k} \times \nabla \psi \right) \right] + \mathbf{k} \cdot \nabla \times (\eta \nabla \chi) - \nabla^2 \left(g h_f + K \right)$$

$$= \mathbf{k} \cdot \left\{ \nabla \eta \times \left(\mathbf{k} \times \nabla \psi \right) + \eta \left[\nabla \times \left(\mathbf{k} \times \nabla \psi \right) \right] \right\}$$

$$+ \mathbf{k} \cdot [\nabla \eta \times \nabla \chi + \eta (\nabla \times \nabla \chi)] - \nabla^2 \left(g h_f + K \right)$$

$$= \mathbf{k} \cdot \left[\mathbf{k} (\nabla \psi \cdot \nabla \eta) - \nabla \psi \left(\nabla \eta \cdot \mathbf{k} \right) \right]$$

$$+ \mathbf{k} \cdot \left\{ \eta \left[\mathbf{k} (\nabla \cdot \nabla \chi) - \nabla \chi \left(\nabla \cdot \mathbf{k} \right) \left(\mathbf{k} \cdot \nabla \right) \nabla \psi + (\nabla \psi \cdot \nabla) \mathbf{k} \right] \right\}$$

$$+ \mathbf{k} \cdot (\nabla \eta \cdot \nabla \chi) - \nabla^2 \left(g h_f + K \right)$$

$$= \nabla \psi \cdot \nabla \eta + \eta \nabla^2 \psi + J(\eta, \chi) - \nabla^2 \left(g h_f + K \right)$$

$$= \nabla \cdot (\eta \nabla \psi) + J(\eta, \chi) - \nabla^2 \left(g h_f + K \right) .$$

The kinetic energy per unit mass, *K,* which appears in (34), can be expressed in terms of the stream function and velocity potential, as follows:

$$K = \frac{\mathbf{v} \cdot \mathbf{v}}{2}$$

$$= \frac{1}{2} \left[\left(\mathbf{k} \times \nabla \psi + \nabla \chi \right) \cdot \left(\mathbf{k} \times \nabla \psi + \nabla \chi \right) \right]$$

$$= \frac{1}{2} \left[\left(\mathbf{k} \times \nabla \psi \right) \cdot \left(\mathbf{k} \times \nabla \psi \right) + \left(\mathbf{k} \times \nabla \chi \right) \cdot \nabla \chi \right.$$

$$\left. + \nabla \chi \cdot \left(\mathbf{k} \times \nabla \psi \right) + \nabla \chi \cdot \nabla \chi \right]$$

$$= \frac{1}{2} \left[\left(\mathbf{k} \times \nabla \psi \right) \cdot \left(\mathbf{k} \times \nabla \psi \right) + \nabla \cdot (\chi \nabla \chi) - \chi \nabla^2 \chi \right]$$

$$+ \nabla \chi \cdot \left(\mathbf{k} \times \nabla \psi \right) \tag{35}$$

$$= \frac{1}{2}\left[\left[\left(\mathbf{k}\times\nabla\psi\right)\times\mathbf{k}\right]\cdot\nabla\psi + \nabla\cdot(\chi\,\nabla\chi) - \chi\,\nabla^2\chi\right] + \mathbf{k}\cdot(\nabla\psi\times\nabla\chi)$$

$$= \frac{1}{2}\left[-\left[\mathbf{k}\times\left(\mathbf{k}\times\nabla\psi\right)\right]\cdot\nabla\psi + \nabla\cdot(\chi\,\nabla\chi) - \chi\,\nabla^2\chi\right] - J(\psi,\,\chi)$$

$$= \frac{1}{2}\left[-\left[\mathbf{k}\left(\nabla\psi\cdot\mathbf{k}\right) - \nabla\psi\left(\mathbf{k}\cdot\mathbf{k}\right)\right]\cdot\nabla\psi + \nabla\cdot(\chi\,\nabla\chi) - \chi\,\nabla^2\chi\right] - J(\psi,\,\chi)$$

$$= \frac{1}{2}\left[\nabla\psi\cdot\nabla\psi + \nabla\cdot(\chi\,\nabla\chi) - \chi\,\nabla^2\chi\right] - J\,(\psi,\,\chi)$$

$$= \frac{1}{2}\left\{\left[\nabla\cdot(\psi\,\nabla\psi) - \psi\,\nabla^2\psi\right] + \nabla\cdot(\chi\,\nabla\chi) - \chi\,\nabla^2\chi\right\} - J(\psi,\,\chi)\ .$$

The Jacobian term of (35) "mixes" the rotational and divergent flow fields. We can show, however, that the integral of the Jacobian over a closed or periodic domain vanishes. This means that the total (volume-integrated, but not mass-weighted) kinetic energy can be expressed in terms of two cleanly separated contributions: a rotational part, and a divergent part.

Finally, the continuity equation can be rewritten in terms of the stream function and velocity potential as follows:

$$\begin{aligned}
\frac{\partial}{\partial t}\,h^* &= -\nabla\cdot(h^*\,\mathbf{v}) \\
&= -\nabla\cdot\left[h^*\left(\mathbf{k}\times\nabla\psi + \nabla\chi\right)\right] \\
&= -\nabla\cdot\left[h^*\left(\mathbf{k}\times\nabla\psi\right)\right] - \nabla\cdot(h^*\,\nabla\chi) \\
&= -\nabla\cdot(h^*\,\nabla\chi) - \left[h^*\,\nabla\cdot\left(\mathbf{k}\times\nabla\psi\right) + \nabla h^*\cdot\left(\mathbf{k}\times\nabla\psi\right)\right] \\
&= -\nabla\cdot(h^*\,\nabla\chi) + \mathbf{k}\cdot(\nabla h^*\times\nabla\psi) \\
&= -\nabla\cdot(h^*\,\nabla\chi) + J(h^*,\,\psi)\ .
\end{aligned} \tag{36}$$

4. Potential vorticity and potential enstrophy

We repeat here for convenience the mass conservation equation [see (24)], i.e.,

$$\frac{\partial}{\partial t} h^* + \nabla \cdot (h^* \mathbf{v}) = 0 \ , \tag{37}$$

and the vorticity conservation equation [see (38)], i.e.,

$$\frac{\partial \eta}{\partial t} + \nabla \cdot (\eta \, \mathbf{v}) = 0 \ . \tag{38}$$

It is interesting that these two equations have exactly the same form. This seems to say that vorticity is somehow like mass!

We define the potential vorticity by

$$q \equiv \frac{\eta}{h^*} \ . \tag{39}$$

Then (39) is immediately equivalent to

$$\frac{\partial}{\partial t} (h^* q) + \nabla \cdot [(h^* \mathbf{v}) q] = 0 \tag{40}$$

which expresses conservation of potential vorticity in flux form. The corresponding advective form can be obtained by subtracting $q \times$ (24) from (40). The result is

$$\frac{\partial q}{\partial t} + (\mathbf{v} \cdot \nabla) q = 0 \ , \tag{41}$$

which says that q is conserved following a particle. Of course, this implies that any function of q is also conserved. In particular, $\frac{1}{2} q^2$ is conserved. This quantity is called the potential enstrophy. We easily find that

$$\frac{\partial}{\partial t} \left(\frac{1}{2} q^2 \right) + (\mathbf{v} \cdot \nabla) \left(\frac{1}{2} q^2 \right) = 0 \tag{42}$$

from which it follows that

$$\frac{\partial}{\partial t} \left[h^* \left(\frac{1}{2} q^2 \right) \right] + \nabla \cdot \left[h^* \mathbf{v} \left(\frac{1}{2} q^2 \right) \right] = 0 \ . \tag{43}$$

5. Energy conservation

The kinetic energy equation can be derived by taking $\mathbf{v}\cdot$ (19), to obtain

$$\frac{\partial K}{\partial t} + \mathbf{v}\cdot\nabla K + \mathbf{v}\cdot\nabla[g(h_S + h^*)] = 0 \ . \tag{44}$$

The last term of (44) represents conversion between kinetic and potential energy. We have used $h_f = h_S + h^*$. We can convert the kinetic energy equation to flux form:

$$\frac{\partial}{\partial t}(h^* K) + \nabla\cdot(\mathbf{v}\, h^* K) + h^*\, \mathbf{v}\cdot\nabla[g(h_S + h^*)] = 0 \ . \tag{45}$$

The potential energy equation can be derived by multiplying (25) by $g(h_S + h^*)$, to obtain

$$\frac{\partial}{\partial t}\left[h^*\left(gh_S + \frac{1}{2}h^*\right)\right] + g(h_S + h^*)\,\nabla\cdot(\mathbf{v}\, h^*) = 0 \ . \tag{46}$$

This can be rewritten as

$$\frac{\partial}{\partial t}\left[h^*\left(gh_S + \frac{1}{2}h^*\right)\right] + \nabla\cdot[\mathbf{v}\, h^*\, g(h_S + h^*)] - h^*\,\mathbf{v}\cdot\nabla[g(h_S + h^*)] = 0 \ . \tag{47}$$

The last term represents conversion between kinetic and potential energy.

If we add (45) and (47), we get

$$\frac{\partial}{\partial t}\left\{h^*\left[K + \left(gh_S + \frac{1}{2}h^*\right)\right]\right\} + \nabla\cdot\{\mathbf{v}\, h^*[K + g(h_S + h^*)]\} = 0 \ . \tag{48}$$

This expresses total energy conservation.