
Reynolds Averaging

David Randall

Introduction

It is neither feasible nor desirable to consider in detail all of the small-scale fluctuations that occur in the atmosphere. For this reason, we introduce averaging or smoothing operators, and attempt to describe only the average state of the atmosphere, following the approach of “Reynolds Averaging,” which takes its name from Osborne Reynolds (Fig. 1), the famous

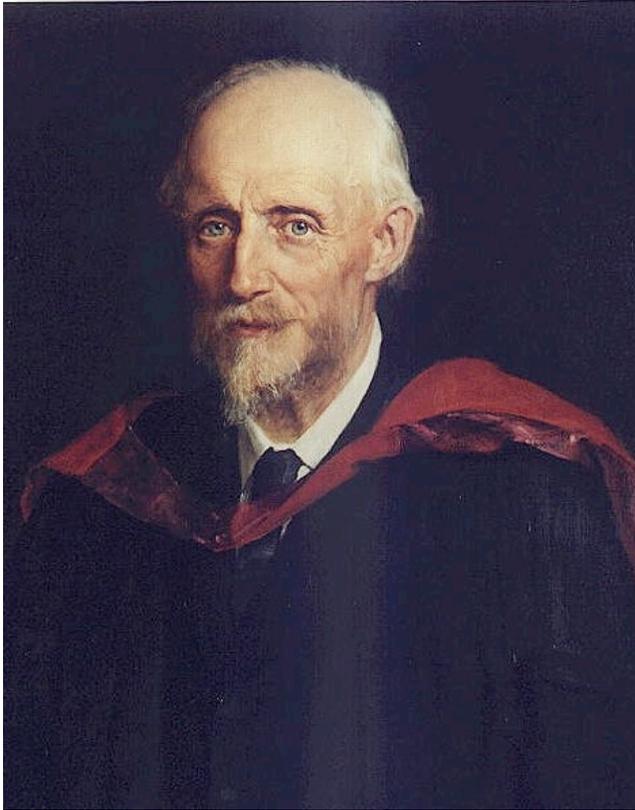


Fig. 1: Osborne Reynolds, after whom Reynolds averaging is named.

aerodynamicist who invented it in the late 19th century (Reynolds, 1895). Averaging leads to new unknowns in the form of additional terms in the governing equations for the averaged quantities. The additional terms represent the effects of “eddy fluxes” that arise from the scales of motion that have been removed the by the averaging procedure.

Depending on the context in which the Reynolds averaging procedure is being used, and the nature of the averaging operator adopted, the eddy fluxes can arise from turbulence, from cumulus convection, from gravity waves, or from departures from zonal uniformity around latitude circles.

The Reynolds conditions

Let u and v be two flow variables (which might or might not be velocity components), and suppose that

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(uv) = S_u, \quad (1)$$

where S_u is a source of u . Let each of the dependent variables of (1) be decomposed as follows:

$$\begin{aligned}
 u &= \bar{u} + u', \\
 v &= \bar{v} + v', \\
 S_u &= \bar{S}_u + S'_u,
 \end{aligned}
 \tag{2}$$

where an overbar denotes an averaging operator that is temporarily undefined. Substitution of (2) into (1) gives

$$\frac{\partial}{\partial t}(\bar{u} + u') + \frac{\partial}{\partial x}(\bar{u} \bar{v} + \bar{u}v' + u'\bar{v} + u'v') = \bar{S}_u + S'_u.
 \tag{3}$$

Now we impose some conditions: We want to *choose* the averaging operator in such a way that the average of (3) reduces to

$$\frac{\partial \bar{u}}{\partial t} + \frac{\partial}{\partial x}(\bar{u} \bar{v} + \overline{u'v'}) = \bar{S}_u.
 \tag{4}$$

In order to arrange this, we require that the average of a sum is the same as the sum of the averages, and that

$$\overline{\left(\frac{\partial u'}{\partial t}\right)} = 0,
 \tag{5}$$

$$\overline{\frac{\partial}{\partial x}(\bar{u}v' + u'\bar{v})} = 0,
 \tag{6}$$

and

$$\overline{S'_u} = 0.
 \tag{7}$$

These requirements have been arrived at by considering a particular and rather simple example. More generally, we require that averaging operators satisfy the “*Reynolds conditions*” (Monin and Yaglom, 1971), which can be stated as follows:

- The average of the sum is the sum of the averages:

$$\overline{f + g} = \bar{f} + \bar{g}.
 \tag{8}$$

- Constants do not affect and are not affected by averaging:

$$\overline{af} = a\overline{f}, \text{ where } a = \text{constant}; \quad (9)$$

$$\overline{a} = a, \text{ where } a = \text{constant}. \quad (10)$$

Properties (8)-(10) together imply that the averaging operator is linear.

- The average of the time or space derivative of a quantity is equal to the corresponding derivative of the average:

$$\overline{\left(\frac{\partial f}{\partial s}\right)} = \frac{\partial \overline{f}}{\partial s}, \text{ where } s \text{ can be either a space coordinate or time;} \quad (11)$$

this enters into (5) and (6).

- The average of the product of an average and an arbitrary function is equal to the product of the averages:

$$\overline{\overline{f}g} = \overline{f} \overline{g}. \quad (12)$$

We did not derive Eqs. (8)-(12). They cannot be derived; they are conditions or requirements that we impose on the averaging operator. What we are striving for here is merely practical simplicity; averaging operators that do not satisfy the Reynolds conditions are just too unruly to be of much use.

From (8)-(12) we can derive the following additional properties:

$$\overline{(\overline{f})} = \overline{f}; \quad (13)$$

$$\overline{f'} = 0; \quad (14)$$

$$\overline{(\overline{f} \overline{g})} = \overline{f} \overline{g}; \quad (15)$$

$$\overline{(\overline{f}g')} = 0; \quad (16)$$

$$\overline{\left(\frac{\partial f'}{\partial s}\right)} = 0, \text{ where } s \text{ can be either a space coordinate or time.} \quad (17)$$

You should confirm for yourself that if the averaging operator satisfies the Reynolds conditions then the average of (3) will reduce to (4).

Alternative averaging operators

Are the conditions (8)-(12) (and the rest) really satisfied? The answer depends on how the averaging operator is defined. Consider three possibilities.

Running averages

Suppose that $\overline{(\quad)}$ is a *running* time average, sometimes called a “smoothing operator:”

$$\bar{f}(t) = \frac{1}{2T} \int_{t-T}^{t+T} f(t^*) w(t^*) dt^* . \quad (18)$$

Here T is a prescribed constant time interval, and $w(t^*)$ is a *weighting function* such that

$\frac{1}{2T} \int_{t-T}^{t+T} w(t^*) dt^* = 1$. The simplest choice is $w(t^*) \equiv 1$. Alternatively, $w(t^*)$ can be a symmetrical

“bell-shaped” function with a maximum at $t^* = t$. Running space averages and combined running space-time averages can be defined in analogous ways.

It should be clear that requirements (8)-(10) are satisfied by (18), because it is a linear operator. What about (12)? For simplicity, suppose that $w(t^*) \equiv 1$. Substituting from (18) into (12), and introducing t^{**} to distinguish between the two integrations, the requirement (12) can be expressed as

$$\frac{1}{2T} \int_{t-T}^{t+T} \left\{ \left[\frac{1}{2T} \int_{t^{**}-T}^{t^{**}+T} f(t^*) dt^* \right] g(t^{**}) \right\} dt^{**} = \left[\frac{1}{2T} \int_{t-T}^{t+T} f(t^*) dt^* \right] \left[\frac{1}{2T} \int_{t-T}^{t+T} g(t^*) dt^* \right]. \quad (19)$$

This equality is *not* generally satisfied because $\frac{1}{2T} \int_{t^{**}-T}^{t^{**}+T} f(t^*) dt^*$ depends on t^{**} through the limits of integration, and so cannot be “pulled out” of the outer integral on the left-hand side of (19). Nevertheless (19) may be approximately satisfied in practice, if T is large enough. In the limit $T \rightarrow \infty$, (19) is satisfied.

What about (11)? Assume again, for simplicity, that $w(t^*) \equiv 1$. Substituting from (18), Eq. (11) becomes

$$\frac{1}{2T} \int_{t-T}^{t+T} \frac{\partial f(t^*)}{\partial t^*} dt^* = \frac{\partial}{\partial t} \left[\frac{1}{2T} \int_{t-T}^{t+T} f(t^*) dt^* \right]. \quad (20)$$

This equality is *not* satisfied because the limits of integration are not constant; using Leibniz' Rule, we can write

$$\frac{\partial}{\partial t} \left[\frac{1}{2T} \int_{t-T}^{t+T} f(t^*) dt^* \right] = \frac{1}{2T} \int_{t-T}^{t+T} \frac{\partial f(t^*)}{\partial t^*} dt^* + \frac{1}{2T} [f(t+T) - f(t-T)]. \quad (21)$$

The term in [] on the right-hand side of (21) is not zero, although it does become small as $T \rightarrow \infty$, which means that we can minimize the problem by letting $T \rightarrow \infty$ with a bell-shaped weighting function of infinite width, e.g., a Gaussian distribution.

In the context of spatial averaging, the issues discussed above are related to the so-called Leonard stress (Leonard, 1974; Germano, 1992).

In case the average is defined as a zonal mean around a latitude circle, as in studies of the global circulation of the atmosphere, the average is taken over the full range of the coordinate (i.e., all the way around), and so the Reynolds conditions are exactly satisfied.

Grid-cell averages

A second approach is to define the averaging operator to be an average over a finite and *fixed* range of the independent variable(s). The simplest way to think about this is to envision a grid of cells, fixed in Eulerian coordinates, as in a discrete numerical model, following the “finite-volume” approach. Because such a model also uses finite time steps, we can define our averaging operator as an average over a four-dimensional “grid cell.”

For simplicity, consider just the discrete “time levels” of the model; averages over such “time cells” can be presented by:

$$\bar{f}(t_1) = \frac{1}{2T} \int_{t_1-T}^{t_1+T} f(t^*) dt^*, \quad (22)$$

where t_1 is the discrete time at the center of the four-dimensional grid cell. Compare (22) with (18). The key difference between (18) and (22) is that *in (22) the limits of integration are constants*, because both t_1 and T are constants.

It should be clear that (8)-(10) are satisfied by (22). The requirement (12) can be expressed as

$$\frac{1}{2T} \int_{t_1-T}^{t_1+T} \left\{ \left[\frac{1}{2T} \int_{t_1-T}^{t_1+T} f(t^*) dt^* \right] g(t^{**}) \right\} dt^{**} = \left[\frac{1}{2T} \int_{t_1-T}^{t_1+T} f(t^*) dt^* \right] \left[\frac{1}{2T} \int_{t_1-T}^{t_1+T} g(t^*) dt^* \right]. \quad (23)$$

This condition is in fact satisfied because $\frac{1}{2T} \int_{t_1-T}^{t_1+T} f(t^*) dt^*$ is independent of time and so can be pulled out of the outer integral on the left-hand side of (23).

Condition (11) is not really applicable in this case, because *averages over finite fixed intervals or “blocks” of the independent variable are not differentiable*. They are not continuous functions of the independent variable. They are discrete. For the case of time averaging, what we get instead of (11) is

$$\left. \frac{\partial f}{\partial t} \right|_{t=t_1} = \frac{1}{2T} \int_{t_1-T}^{t_1+T} \frac{\partial f}{\partial t^*} dt^* = \frac{f(t_1+T) - f(t_1-T)}{2T}. \quad (24)$$

Ensemble averages

A third possibility is to define $\overline{(\quad)}$ to be an average over an infinite ensemble of realizations. This means that we imagine that we measure u and v for each of an infinity of experiments that are “alike” except in the details of their turbulence, and obtain \bar{u} and \bar{v} by averaging over the ensemble. Suppose that we number the experiments by assigning to each a value of the parameter r , which we can call the “realization coordinate,” and suppose further that an especially diligent graduate student carries out an experiment for each point on the infinite real number line. In this case, averaging over the ensemble is equivalent to averaging over the coordinate r , from $-\infty$ to ∞ . Because our fluid dynamical equations do not involve differentiation with respect to r , there are no difficulties of the sort exemplified by (21). Because our average is over all values of r , \bar{u} is independent of r , and so (12) is clearly satisfied. Further discussion is given by Monin and Yaglom (1971, pp. 205-222).

It goes without saying that, in practice, turbulence data are not averaged over an infinity of realizations. In laboratory settings, e.g., the study of turbulence in a wind tunnel, it is possible to obtain data that can be used to compute ensemble averages over a finite number of realizations that are “the same” except for presumably unimportant details. In studies of the uncontrolled atmosphere, however, it is difficult if not impossible to obtain multiple realizations of “the same” situation; the best that we can do is to simulate multiple realizations of the same situation using high-resolution models with slightly different initial conditions (e.g., Moeng, 1986; Krueger 1988), and compute averages from the ensemble of simulations. The size of such an ensemble is obviously finite.

Dependence of the solution on grid size

Suppose that we apply Reynolds-averaged equations in a numerical model of the atmosphere that is used to forecast or simulate the distributions of various prognostic and diagnostic variables over the grid. For the free atmosphere, the subgrid-scale fluxes might represent the vertical exchanges associated with cumulus convection. In a model with a grid too coarse to resolve cumulus convection, such fluxes would be determined using a cumulus parameterization.

In such a model, it is natural to define the averaging operator in terms of *spatial averages* over the grid cells; this is the finite-volume method. In case the horizontal grid size of the model is on the order of 100 km or larger, we can imagine that each grid cell contains many individual cumulus clouds, so that the grid-cell averages can be regarded as representing the collective effects of the many clouds co-existing inside the grid cell at a given time. If we now imagine reducing the grid size of the model to a relatively small value, e.g., 1 km, the spatial averages over individual grid cells should reflect the presence of the larger individual cumuli, which would be marginally represented on such a grid.

Note, however, that there is no reason why the averaging distance has to be the same as the grid size. We could choose to associate the averaging distance with a physical length scale, which should be independent of the somewhat arbitrarily chosen grid size. In particular, the averaging distance can be larger than the grid size. In such a case, we can imagine reducing the grid size without changing the averaging length. Solutions should become smooth as the grid scale becomes much finer than the averaging length. With a smooth solution, the fine resolution is wasted, however.

Finally, suppose that we define the averaging operator as an ensemble average. As the grid size is reduced, the ensemble average will remain spatially and temporally smooth, because the individual cumuli occur in different places and at different times in the different realizations that make up the ensemble. The individual members of the ensemble would produce individual cumulus clouds in different places at any given time, but the large-scale spatial average of a variable, e.g., the precipitation rate, would presumably be approximately the same for all members of the ensemble. This means that for the case of large grid cells the spatial average and the ensemble average are approximately equal.

Summary and conclusions

Reynolds averaging is used in practically all areas of fluid dynamics, and in particular it finds very diverse and important applications in geophysical fluid dynamics. The averaging operator can be defined in a variety of ways, each of which has some advantages and disadvantages.

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