Reynolds Averaging

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1. Introduction

It is neither feasible nor desirable to consider in detail all of the small-scale fluctuations that occur in the atmosphere. For this reason, we introduce averaging or smoothing operators, and attempt to describe only the average state of the atmosphere, following the approach of "Reynolds Averaging." This leads, however, to additional terms in the governing equations for the averaged quantities; these additional terms represent the effects of "eddy fluxes" that arise from the scales of motion that have been removed the by the averaging procedure.

Depending on the context in which the Reynolds averaging procedure is being used and the nature of the averaging operator adopted, the eddy fluxes can arise from turbulence, from cumulus convection, from gravity waves, or, in large-scale models, from departures from zonal uniformity around latitude circles.

2. The Reynolds conditions

Let u and v be two flow variables (perhaps velocity components), and suppose that

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(uv) = S_u \ , \tag{1}$$

where S_u is a source/sink of u, whose exact nature we need not specify here. Let each of the dependent variables of (1) be decomposed as follows:

$$\begin{aligned} u &= \bar{u} + u' \ , \\ v &= \bar{v} + v' \ , \\ S_u &= \bar{S_u} + S_u' \ , \end{aligned} \tag{2}$$

where the bar indicates an averaging operator which is temporarily undefined. Substitution of (2) into (1) gives

$$\frac{\partial}{\partial t}\left(\bar{u}+u'\right)+\frac{\partial}{\partial x}\left(\bar{u}\,\bar{v}+\bar{u}\,v'+u'\,\bar{v}+u'\,v'\right)=\bar{S}_u+S' \quad . \tag{3}$$

We choose the averaging operator in such a way that the average of (3) is

$$\frac{\partial \bar{u}}{\partial t}+\frac{\partial}{\partial x}\left(\bar{u}\,\bar{v}+\overline{u'\,v'}\right)=\bar{S}'_u \quad . \tag{4}$$

That is, we require that

$$\overline{\left(\frac{\partial u'}{\partial t}\right)}=0 \quad . \tag{5}$$

$$\overline{\frac{\partial}{\partial x}\left(\bar{u}\,v'+u'\,\bar{v}\right)}=0\,, \tag{6}$$

and

$$\overline{S'_u}=0 \quad . \tag{7}$$

These requirements have been arrived at by considering a particular and rather simple example. More generally, we require that averaging operators meet the “Reynolds conditions” [Monin and Yaglom (1971), which can be stated as follows:

- The average of the sum is the sum of the averages:

$$\overline{f+g}=\bar{f}+\bar{g} \quad . \tag{8}$$

- Constants do not affect and are not affected by averaging:

$$\overline{af}=a\,\bar{f}, \text{ where } a = \text{ constant;} \tag{9}$$

$$\bar{a}=a \text{ where } a = \text{constant.} \tag{10}$$

Properties (8)-(10) together imply that the averaging operator is “linear.”

- The average of the time or space derivative of a quantity is equal to the corresponding derivative of the average:

$$\overline{\left(\frac{\partial f}{\partial s}\right)}=\frac{\partial \bar{f}}{\partial s}, \text{ where } s \text{ can be either } a \text{ space coordinate or time;} \tag{11}$$

this enters into (5) and (6).

- The average of the product of an average and a function is equal to the product of the averages:

$$\overline{\overline{f}\,g} = \overline{f}\,\bar{g}, \tag{12}$$

Eqs. (8)-(12) have not been and cannot be *derived*; they are *requirements* that we impos on the averaging operator. What we are striving for here is nothing more or less than practical simplicity; averaging operators that do not satisfy the Reynolds conditions are just too unruly to be of much use. From (8)-(12) we can derive the following additiona properties:

$$\overline{\overline{f}} = \overline{f}\ ; \tag{13}$$

$$\overline{f'} = 0\ ; \tag{14}$$

$$\overline{\overline{f}\,\overline{g}} = \overline{f}\,\overline{g}\ ; \tag{15}$$

$$\overline{f}\,g' = \overline{f}\,g' \tag{16}$$

$$\overline{\left(\frac{\partial f'}{\partial s}\right)} = 0\ ,\ \text{where } s \text{ can be either a space coordinate or time.} \tag{17}$$

3. Alternative averaging operators

Can the requirements (8)-(12) (and the rest) really be met? The answer depends o how the averaging operator is defined. Consider three possibilities.

3.1. Running averages.

Suppose that $\overline{(\,)}$ is a running time average, sometimes called a "smoothing operator:"

$$\overline{f}(t) = \frac{1}{2T}\int_{t-T}^{t+T} f(t^*)\,dt^*\ . \tag{19}$$

Here T is a prescribed constant time interval, and t^* is a dummy variable of integration. Running space averages and combined running space-time averages can be defined in analogous ways. Spatial averages are technically difficult to measure; so, in practice, averaging is almost always done over a finite time interval. As a result, (4), (5), and (6) are not exactly satisfied, but good accuracy can often be achieved, provided that T is much longer than the typical time scales for fluctuations of u'. Note that if u' does not fluctuate, then it is zero, so that (4), (5), and (6) are trivially satisfied.

It should be clear that requirements (8)-(10) are satisfied by (19), which is a line operator. What about (12)? Substituting, and introducing t^{**} to distinguish between the two integrations, we find that our requirement (11) can be written as

$$\frac{1}{2T}\int_{t-T}^{t+T}\left\{\left[\frac{1}{2T}\int_{t^{**}-T}^{t^{**}+T} f(t^*)\,dt^*\right]g(t^{**})\right\}dt^{**}$$

$$=\left[\frac{1}{2T}\int_{t^{**}-T}^{t^{**}+T} f(t^*)\,dt^*\right]\left[\frac{1}{2T}\int_{t^{**}-T}^{t^{**}+T} g(t^*)\,dt^*\right]. \tag{20}$$

This requirement is *not* generally satisfied because $\overline{f} \equiv \frac{1}{2T}\int_{t^{**}-T}^{t^{**}+T} f(t^*)\,dt^*$ depends on t^{**}, and so cannot be “pulled out” of the outer integral on the left-hand side of (20). Th problem arises because in a running time average the limits of integration depend on time. Nevertheless (20) is often nearly satisfied in practice.

What about (11)? Using (19), we can express (11) (for the case) as

$$\frac{1}{2T}\int_{t-T}^{t+T}\frac{\partial f(t^*)}{\partial t^*}\,dt^* = \frac{\partial}{\partial t}\left[\frac{1}{2T}\int_{t-T}^{t+T} f(t^*)\,dt^*\right]. \tag{21}$$

This equality is *not* satisfied because the limits of integration are not constant; using Leibniz’ Rule, we can write

$$\frac{\partial}{\partial t}\left[\frac{1}{2T}\int_{t-T}^{t+T} f(t^*)\,dt^*\right] = \frac{1}{2T}\int_{t-T}^{t+T}\frac{\partial f(t^*)}{\partial t^*}\,dt^* + \frac{1}{2T}\left[f(t+T)-f(t-T)\right]. \tag{22}$$

The term in [ ] on the right-hand side of (22) is not zero, although it does become small a $T \to \infty$.

We conclude that running averages do not satisfy the Reynolds conditions.

We can get around these problems by letting $T \to \infty$, but if we do so we lose all ability to predict the temporal behavior of the mean state — not a very satisfactory solution!

3.2. *Grid cell averages.*

A second possibility is to define $\overline{(\ )}$ to be an average over a finite and fixed range of the independent variable(s). The simplest way to think about this is to envision a grid of cells, fixed in Eulerian coordinates, as in a finite-difference model. Keeping in mind that such a model also uses finite time steps, we can define our averaging operator as an average over a four-dimensional "grid cell."

For simplicity, consider just the discrete "time levels" of the model; averages over such "time cells" can be presented by:

$$\overline{f}(t_1) = \frac{1}{2T}\int_{t_1-T}^{t_1+T} f(t^*)\,dt^* \ , \tag{23}$$

where t_1 is the discrete time at the center of the four-dimensional grid cell. Compare (23) with (19). The key difference between (19) and (23) is that in (23) the limits of integration are constants, because both t_1 and T are constants.

It should be clear that (8)-(10) are satisfied by (23). Eq. (12) becomes

$$\begin{aligned}&\frac{1}{2T}\int_{t_1-T}^{t_1+T}\left\{\left[\frac{1}{2T}\int_{t_1-T}^{t_1+T} f(t^*)\,dt^*\right]g(t^{**})\right\}dt^{**}\\ &=\left[\frac{1}{2T}\int_{t_1-T}^{t_1+T} f(t^*)\,dt^*\right]\left[\frac{1}{2T}\int_{t_1-T}^{t_1+T} g(t^*)\,dt^*\right] .\end{aligned} \tag{24}$$

This condition is in fact satisfied because $\left[\frac{1}{2T}\int_{t_1-T}^{t_1+T} f(t^*)\,dt^*\right]$ is independent of time and so can be pulled out of the outer integral on the left-hand side of (24). Similarly, (11) becomes

$$\frac{1}{2T}\int_{t_1-T}^{t_1+T}\frac{\partial f(t^*)}{\partial t^*}\,dt^* = \frac{\partial}{\partial t}\left[\frac{1}{2T}\int_{t_1-T}^{t_1+T} f(t^*)\,dt^*\right] , \tag{25}$$

which is satisfied because the limits of integration on the right-hand side of (25) are constants. The same conclusions hold for four-dimensional averages over four-dimensional grid cells.

This demonstrates that grid-cell averages can satisfy the requirements (8)-(12) without problems. On the other hand, the grid-averaged quantities are defined only at discrete points in space and time, and so are not spatially or temporally differentiable. In some applications this can be a disadvantage.

3.3. *Ensemble averages.*

A third possibility is to define $\overline{(\ )}$ to be an average over an infinite ensemble of realizations. This means that we imagine that we measure u and v for each of an infinity of experiments which are "alike" except in the details of their turbulence, and obtain u and v by averaging over the ensemble. Suppose that we number the experiments by assigning to each a value of the parameter $\hat{v}$, which we can call the "realization coordinate," and suppose further that an especially diligent graduate student carries out an experiment for each point on the infinite real number line. In this case, averaging over the ensemble is equivalent to averaging over the coordinate $\hat{v}$, from $-\infty$ to ∞. Because our fluid dynamical equations do not involve differentiation with respect to $\hat{v}$, we encounter no difficulties of the sort exemplified by (22). Because our average is over all values of $\hat{v}$, u is independent of $\hat{v}$, and so (12) is clearly satisfied. Further discussion is given by [Monin and Yaglom (1971), pp. 205-222].

It goes without saying that, in practice, turbulence data is not averaged over an infinity of realizations. In laboratory settings, e.g., the study of turbulence in a wind tunnel, it is possible to obtain data that can be used to compute ensemble averages over a finite number of realizations which are "the same" except for presumably unimportant details. In studies of the uncontrolled atmosphere, however, it is difficult if not impossible to obtain multiple realizations of "the same" situation; the best that we can do is to produce multiple simulations of the same situation using high-resolution models [e.g., Moeng (1986); Kreuger (1988)], and compute averages from the ensemble of simulations.

4. Dependence of the solution on grid size

Suppose that we apply Reynolds-averaged equations in a numerical model of the atmosphere that is used to forecast or simulate the distributions of various prognostic and diagnostic variables over the grid. For the free atmosphere, the subgrid-scale fluxes might represent the vertical exchanges associated with cumulus convection; such fluxes would be determined using a cumulus parameterization.

Suppose that we define the averaging operator in terms of *spatial averages* over the grid cells. In case the horizontal grid size of the model is on the order of 100 km or larger, we can imagine that each grid cell contains many individual cumulus clouds, so that the grid-cell averages can be regarded as representing the collective effects of the many clouds co-existing inside the grid cell at a given time. If we now imagine reducing the grid size of the model to a relatively small value, e.g., 1 km, the spatial averages over individual grid cells should reflect the presence of the larger individual cumuli, which would be marginally represented on such a grid[1]. It is difficult to imagine a "cumulus parameterization" that could produce realistic area-averages as the grid size is varied from 100 km to 1 km.

Note, however, that there is no reason why the averaging distance has to be the same as the grid size. We can choose to associate the averaging distance with a physical length scale, which should be independent of the somewhat arbitrarily chosen grid size. In particular, the averaging distance can be larger than the grid size. In such a case, we can imagine reducing the grid size without changing the averaging length. Solutions would be expected to become smooth as the grid scale becomes much finer than the averaging length.

Finally, suppose that we define the averaging operator as an ensemble average. As the grid size is reduced, the ensemble average will remain spatially and temporally smooth, because the individual cumuli occur in different places and at different times in the various realizations that make up the ensemble. The individual members of the ensemble would produce individual cumulus clouds in different places at any given time, but the average of a variable, e.g., the precipitation rate, over the large grid-cell area would presumably be approximately the same for all members of the ensemble. This means that for the case of large grid cells the spatial average and the ensemble average are approximately equal.

5. Summary and conclusions

Reynolds averaging is used in practically all areas of fluid dynamics, and in particular it finds very wide applications in geophysical fluid dynamics. The averaging operator can be defined in a variety of ways, each of which has some advantages and disadvantages. In a model with highly variable spatial resolution, ensemble averaging has important conceptual advantages over spatial averaging.

Notes

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[1] Note that the model with sufficiently high horizontal resolution has to take into account the divergences of the *horizontal* eddy fluxes associated with cumulus convection and other eddies.

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