Introduction

It is neither feasible nor desirable to consider in detail all of the small-scale fluctuations that occur in the atmosphere. For this reason, we introduce averaging or smoothing operators, and attempt to describe only the average state of the atmosphere, following the approach of “Reynolds Averaging.” This leads, however, to additional terms in the governing equations for the averaged quantities. The additional terms represent the effects of “eddy fluxes” that arise from the scales of motion that have been removed by the averaging procedure.

Depending on the context in which the Reynolds averaging procedure is being used, and the nature of the averaging operator adopted, the eddy fluxes can arise from turbulence, from cumulus convection, from gravity waves, or from departures from zonal uniformity around latitude circles.

The Reynolds conditions

Let $u$ and $v$ be two flow variables (which might be velocity components), and suppose that

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(uv) = S_u,$$

(1)

where $S_u$ is a source of $u$. Let each of the dependent variables of (1) be decomposed as follows:

$$u = \bar{u} + u',$$
$$v = \bar{v} + v',$$
$$S_u = \bar{S}_u + S'_u,$$

(2)

where an overbar indicates a linear averaging operator that is temporarily undefined. Substitution of (2) into (1) gives
\[
\frac{\partial}{\partial t} (\bar{u} + u') + \frac{\partial}{\partial x} (\bar{u} \bar{v} + \bar{u}v' + u'\bar{v} + u'v') = \bar{S}_u + S'.
\]  
(3)

We want to choose the averaging operator in such a way that the average of (3) is

\[
\frac{\partial \bar{u}}{\partial t} + \frac{\partial}{\partial x} (\bar{u} \bar{v} + u'\bar{v}') = \bar{S}_u'.
\]  
(4)

In order to arrange this, we require that

\[
\frac{\partial}{\partial t} \left( \frac{\partial u'}{\partial t} \right) = 0,
\]  
(5)

\[
\frac{\partial}{\partial x} (\bar{u}v' + u'\bar{v}) = 0,
\]  
(6)

and

\[
\bar{S}'_u = 0.
\]  
(7)

These requirements have been arrived at by considering a particular and rather simple example. More generally, we require that averaging operators meet the “Reynolds conditions” (Monin and Yaglom, 1971), which can be stated as follows:

- The average of the sum is the sum of the averages:
  \[
  \bar{f} + \bar{g} = \bar{f} + \bar{g}.
  \]  
  (8)

- Constants do not affect and are not affected by averaging:
  \[
  \bar{af} = \bar{af}, \text{ where } a = \text{ constant} ;
  \]  
  (9)

  \[
  \bar{a} = \bar{a}, \text{ where } a = \text{ constant} .
  \]  
  (10)

Properties (8)-(10) together imply that the averaging operator is “linear.”

- The average of the time or space derivative of a quantity is equal to the corresponding derivative of the average:
\[
\left( \frac{\partial \bar{f}}{\partial s} \right) = \frac{\partial \bar{f}}{\partial s}, \quad \text{where } s \text{ can be either a space coordinate or time;}
\]

(11)

this enters into (5) and (6).

- The average of the product of an average and a function is equal to the product of the averages:

\[
\bar{f} g = \bar{f} \bar{g}.
\]

(12)

Eqs. (8)-(12) have not been and cannot be derived; they are requirements that we impose on the averaging operator. What we are striving for here is merely practical simplicity; averaging operators that do not satisfy the Reynolds conditions are just too unruly to be of much use.

From (8)-(12) we can derive the following additional properties:

\[
\bar{f} = \bar{f};
\]

(13)

\[
\bar{f}' = 0;
\]

(14)

\[
\bar{f} \bar{g} = \bar{f} \bar{g};
\]

(15)

\[
\bar{f} \bar{g}' = 0;
\]

(16)

\[
\left( \frac{\partial \bar{f}'}{\partial s} \right) = 0, \quad \text{where } s \text{ can be either a space coordinate or time.}
\]

(17)

**Alternative averaging operators**

Are the requirements (8)-(12) (and the rest) really satisfied? The answer depends on how the averaging operator is defined. Consider three possibilities.

**Running averages**

Suppose that \( \bar{ } \) is a running time average, sometimes called a “smoothing operator:”
\[ f(t) = \frac{1}{2T} \int_{t-T}^{t+T} f(t^*) w(t^*) dt^*. \]  

(18)

Here \( T \) is a prescribed constant time interval, and \( w(t^*) \) is a weighting function such that \( \frac{1}{2T} \int_{t-T}^{t+T} w(t^*) dt^* = 1 \). The simplest choice is \( w(t^*) \equiv 1 \). Alternatively, \( w(t^*) \) can be a symmetrical “bell-shaped” function with a maximum at \( t^* = t \). Running space averages and combined running space-time averages can be defined in analogous ways. Spatial averages are technically difficult to measure, so, in practice, averaging is almost always done over a finite time interval, \( T \), which might typically be on the order of half an hour. As a result, (4), (5), and (6) are not exactly satisfied, but good accuracy can often be achieved, provided that \( T \) is much longer than the typical time scales for fluctuations of \( u' \). Note that if \( u' \) does not fluctuate, then it is zero, so that (4), (5), and (6) are trivially satisfied.

It should be clear that requirements (8)-(10) are satisfied by (18), which is a linear operator. What about (12)? For simplicity, suppose that \( w(t^*) \equiv 1 \). Substituting from (18), \( Eq. (11) \) becomes

\[ \frac{1}{2T} \int_{t-T}^{t+T} \left[ \frac{1}{2T} \int_{t-T}^{t+T} f(t^*) dt^* \right] g(t^*) dt^* = \frac{1}{2T} \int_{t-T}^{t+T} g(t^*) dt^* \left[ \frac{1}{2T} \int_{t-T}^{t+T} f(t^*) dt^* \right]. \]  

(19)

This equality is not generally satisfied because \( f \equiv \frac{1}{2T} \int_{t-T}^{t+T} f(t^*) dt^* \) depends on \( t^{**} \), and so cannot be “pulled out” of the outer integral on the left-hand side of (19). The problem arises because in a running time average the limits of integration depend on time. Nevertheless (19) may be nearly satisfied in practice, if \( T \) is large enough. In the limit \( T \to \infty \), (19) is satisfied.

What about (11)? Again we assume for simplicity that \( w(t^*) \equiv 1 \). Substituting from (18), Eq. (11) becomes

\[ \frac{1}{2T} \int_{t-T}^{t+T} \frac{df(t^*)}{dt^*} dt^* = \frac{\partial}{\partial t} \left[ \frac{1}{2T} \int_{t-T}^{t+T} f(t^*) dt^* \right]. \]  

(20)

This equality is not satisfied because the limits of integration are not constant; using Leibniz’ Rule, we can write
\[
\frac{\partial}{\partial t} \left[ \frac{1}{2T} \int_{t-T}^{t+T} f(t^*) dt^* \right] = \frac{1}{2T} \int_{t-T}^{t+T} \frac{\partial f(t^*)}{\partial t} dt^* + \frac{1}{2T} [f(t+T) - f(t-T)].
\]

(21)

The term in [ ] on the right-hand side of (21) is not zero, although it does become small as \( T \to \infty \), which means that we can get around the problem by letting \( T \to \infty \) with a bell-shaped weighting function of infinite width, e.g., a Gaussian distribution.

In case the average is defined as a zonal mean around a latitude circle, as in studies of the general circulation of the atmosphere, the average is taken over the full range of the coordinate (i.e., all the way around), and so the Reynolds conditions are exactly satisfied. The condition (11) takes the trivial form \( 0 = 0 \) in this case, however.

**Grid-cell averages**

A second approach is to define the averaging operator to be an average over a finite and fixed range of the independent variable(s). The simplest way to think about this is to envision a grid of cells, fixed in Eulerian coordinates, as in a finite-difference model, following the “finite-volume” approach. Keeping in mind that such a model also uses finite time steps, we can define our averaging operator as an average over a four-dimensional “grid cell.”

For simplicity, consider just the discrete “time levels” of the model; averages over such “time cells” can be presented by:

\[
\bar{f}(t_i) = \frac{1}{2T} \int_{t-i}^{t+i} f(t^*) dt^*,
\]

(22)

where \( t_i \) is the discrete time at the center of the four-dimensional grid cell. Compare (22) with (18). The key difference between (18) and (22) is that in (22) the limits of integration are constants, because both \( t_i \) and \( T \) are constants.

It should be clear that (8)-(10) are satisfied by (22). Eq. (12) becomes

\[
\frac{1}{2T} \int_{t_i-T}^{t_i+T} \left[ \frac{1}{2T} \int_{t_i-T}^{t_i+T} f(t^*) dt^* \right] g(t^*) \left\{ \int_{t_i-T}^{t_i+T} f(t^*) dt^* \right\} dt = \frac{1}{2T} \int_{t_i-T}^{t_i+T} f(t^*) dt^* \left\{ \frac{1}{2T} \int_{t_i-T}^{t_i+T} g(t^*) dt^* \right\}.
\]

(23)

This condition is in fact satisfied because \( \frac{1}{2T} \int_{t_i-T}^{t_i+T} f(t^*) dt^* \) is independent of time and so can be pulled out of the outer integral on the left-hand side of (23).
Condition (11) is not really applicable in this case, because *averages over finite fixed intervals of the independent variable are not differentiable.* They are not continuous functions of the independent variable.

We conclude that grid-cell averages can satisfy the Reynolds conditions, except for (11), without problems.

**Ensemble averages**

A third possibility is to define \( \langle \rangle \) to be an average over an infinite ensemble of realizations. This means that we imagine that we measure \( u \) and \( v \) for each of an infinity of experiments that are “alike” except in the details of their turbulence, and obtain \( \bar{u} \) and \( \bar{v} \) by averaging over the ensemble. Suppose that we number the experiments by assigning to each a value of the parameter \( r \), which we can call the “realization coordinate,” and suppose further that an especially diligent graduate student carries out an experiment for each point on the infinite real number line. In this case, averaging over the ensemble is equivalent to averaging over the coordinate \( r \), from \(-\infty\) to \(\infty\). Because our fluid dynamical equations do not involve differentiation with respect to \( r \), we encounter no difficulties of the sort exemplified by (21). Because our average is over all values of \( r \), \( \bar{u} \) is independent of \( r \), and so (12) is clearly satisfied. Further discussion is given by Monin and Yaglom (1971, pp. 205-222).

It goes without saying that, in practice, turbulence data are not averaged over an infinity of realizations. In laboratory settings, e.g., the study of turbulence in a wind tunnel, it is possible to obtain data that can be used to compute ensemble averages over a finite number of realizations that are “the same” except for presumably unimportant details. In studies of the uncontrolled atmosphere, however, it is difficult if not impossible to obtain multiple realizations of “the same” situation; the best that we can do is to produce multiple simulations of the same situation using high-resolution models (e.g., Moeng, 1986; Krueger 1988), and compute averages from the ensemble of simulations. The size of such an ensemble is obviously finite, however.

**Dependence of the solution on grid size**

Suppose that we apply Reynolds-averaged equations in a numerical model of the atmosphere that is used to forecast or simulate the distributions of various prognostic and diagnostic variables over the grid. For the free atmosphere, the subgrid-scale fluxes might represent the vertical exchanges associated with cumulus convection. In a low-resolution model, such fluxes would be determined using a cumulus parameterization.

Suppose that we define the averaging operator in terms of *spatial averages* over the grid cells. In case the horizontal grid size of the model is on the order of 100 km or larger, we can imagine that each grid cell contains many individual cumulus clouds, so that the grid-cell averages can be regarded as representing the collective effects of the many clouds co-existing inside the grid cell at a given time. If we now imagine reducing the grid size of the model to a
relatively small value, e.g., 1 km, the spatial averages over individual grid cells should reflect the presence of the larger individual cumuli, which would be marginally represented on such a grid.

Note, however, that there is no reason why the averaging distance has to be the same as the grid size. We can choose to associate the averaging distance with a physical length scale, which should be independent of the somewhat arbitrarily chosen grid size. In particular, the averaging distance can be larger than the grid size. In such a case, we can imagine reducing the grid size without changing the averaging length. Solutions should become smooth as the grid scale becomes much finer than the averaging length. With a smooth solution, the fine resolution is wasted, however.

Finally, suppose that we define the averaging operator as an ensemble average. As the grid size is reduced, the ensemble average will remain spatially and temporally smooth, because the individual cumuli occur in different places and at different times in the various realizations that make up the ensemble. The individual members of the ensemble would produce individual cumulus clouds in different places at any given time, but the large-scale spatial average of a variable, e.g., the precipitation rate, would presumably be approximately the same for all members of the ensemble. This means that for the case of large grid cells the spatial average and the ensemble average are approximately equal.

Summary and conclusions

Reynolds averaging is used in practically all areas of fluid dynamics, and in particular it finds very wide applications in geophysical fluid dynamics. The averaging operator can be defined in a variety of ways, each of which has some advantages and disadvantages. In a model with highly variable spatial resolution, ensemble averaging has important conceptual advantages over spatial averaging.

References and Bibliography

