

Conservation of momentum on a rotating sphere

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26.1 Conservation of momentum on a rotating sphere

Newton's statement of momentum conservation, as applied in an inertial (i.e. non-accelerating) frame of reference, can be written as follows:

$$\frac{D_a \mathbf{V}_a}{Dt} = -\nabla\phi_a - \alpha\nabla p - \alpha\nabla \bullet \mathsf{F} - \alpha\nabla p \; . \qquad (26.1)$$

Here $\frac{D_a(\;)}{Dt}$ is the Lagrangian derivative in the inertial frame, and $\mathbf{V}_a$ is the velocity as seen in the inertial frame. The left-hand-side of (26.1) represents the acceleration of the air as seen in an inertial frame. The precise meaning of the acceleration $\frac{D_a \mathbf{V}_a}{Dt}$ is clarified below. The gravitational potential is ϕ_a, and F is the stress tensor associated with molecular effects. Note that $\nabla \bullet \mathsf{F}$ is a vector.

The length of a day is 86400 s, so the Earth rotates about its axis with an angular velocity of $\frac{2\pi}{(86400 \text{ s})} = 7.29 \times 10^{-5} s^{-1}$. This angular velocity can be represented by a vector, $\mathbf{\Omega}$, pointing towards the celestial North Pole. Consider a coordinate system that is rotating with the Earth, and refer to Fig. 26.1. A rotating coordinate system is not an inertial frame, so Eq. (26.1) must be transformed to describe momentum conservation in the rotating frame. Let $\mathbf{r}$ be a position vector extending from the center of the Earth to a particle of air whose position is generally changing with time. The "absolute velocity" of the air, $\mathbf{V}_a \equiv \frac{D_a \mathbf{r}}{Dt}$, is related to its relative velocity, $\mathbf{V} \equiv \frac{D\mathbf{r}}{Dt}$, as seen in the rotating coordinate system, by

$$\frac{D_a \mathbf{r}}{Dt} = \frac{D\mathbf{r}}{Dt} + \mathbf{\Omega} \times \mathbf{r} \; , \qquad (26.2)$$

or

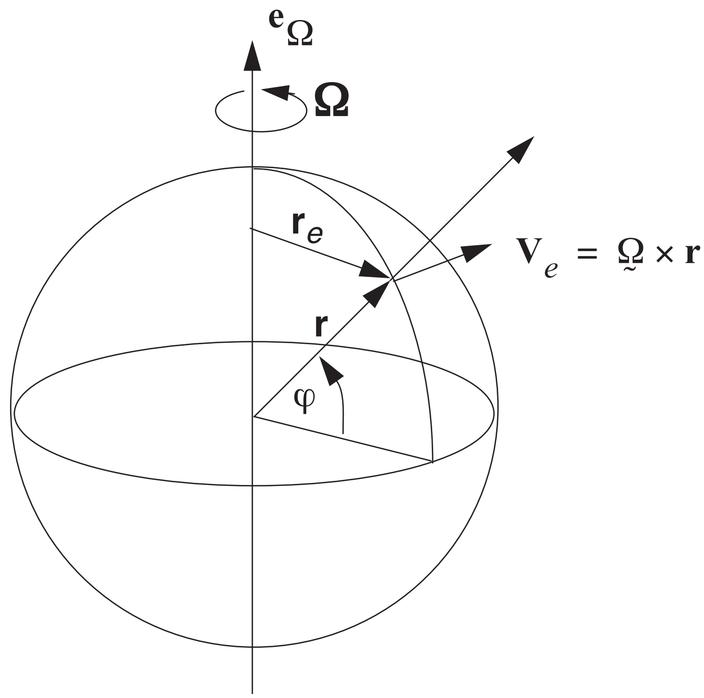

Figure 26.1: Sketch defining vectors used in the text.

$$\mathbf{V}_a = \mathbf{V} + \mathbf{\Omega} \times \mathbf{r} . \tag{26.3}$$

Because in general the air is moving with respect to the Earth, $\mathbf{r}$ is changing with time as seen in the rotating frame. According to (26.3), the velocity as seen in the rotating frame is different from the velocity as seen in the inertial frame. A transformation of the form (26.2) can be applied to *any vector*; here we have applied it to the position vector, $\mathbf{r}$.

The acceleration in the inertial frame is related to the acceleration in the rotating frame by

$$\frac{D_a \mathbf{V}_a}{Dt} = \frac{D\mathbf{V}_a}{Dt} + \mathbf{\Omega} \times \mathbf{V}_a . \tag{26.4}$$

Note that (26.2) and (26.4) have the same form. According to (26.4), the acceleration in the rotating frame is different from the acceleration in the inertial frame. Substituting for $\mathbf{V}_a$, from (26.3), we obtain

$$\frac{D_a \mathbf{V}_a}{Dt} = \frac{D}{Dt}(\mathbf{V} + \mathbf{\Omega} \times \mathbf{r}) + \mathbf{\Omega} \times (\mathbf{V} + \mathbf{\Omega} \times \mathbf{r})$$

$$= \frac{D\mathbf{V}}{Dt} + 2\mathbf{\Omega} \times \mathbf{V} + \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r}) \ . \tag{26.5}$$

This relates the absolute acceleration, $\frac{D_a \mathbf{V}_a}{Dt}$, to the apparent acceleration as seen in the rotating frame, i.e. $\frac{D\mathbf{V}}{Dt}$.

Using (26.5) in (26.1), we find that the equation of motion relative to the rotating frame is

$$\frac{D\mathbf{V}}{Dt} = -2\mathbf{\Omega} \times \mathbf{V} - \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r}) - \nabla\phi_a - \alpha\nabla p - \alpha\nabla \bullet \mathbf{F} . \tag{26.6}$$

The term $-2\mathbf{\Omega} \times \mathbf{V}$ is the Coriolis acceleration, whose direction is perpendicular to $\mathbf{V}$. The term $-\mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r})$ is the centripetal acceleration. You should be able to show that

$$\mathbf{V}_e = \mathbf{\Omega} \times \mathbf{r} = (\Omega r \cos\varphi)\mathbf{e}_\lambda , \tag{26.7}$$

where $\mathbf{e}_\lambda$ is a unit vector pointing east, φ is latitude, and $\mathbf{V}_e$ is the velocity (as seen in the inertial frame) that a particle at radius r and latitude φ experiences due to the Earth's rotation (refer to Fig. 26.1). With this notation, we find that

$$-\mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r}) = (\Omega^2 r \cos\varphi)\mathbf{e}_\lambda \times \mathbf{e}_\Omega$$

$$= \Omega^2 \mathbf{r}_e \quad , \tag{26.8}$$

where $\mathbf{r}_e$ is the vector shown in Fig. 26.1, and $\mathbf{e}_\Omega$ is a unit vector pointing toward the celestial north pole. This shows that the centripetal acceleration points outward, in the direction of $\mathbf{r}_e$, which is perpendicular to the axis of the Earth's rotation. It can be shown that

$$\Omega^2 \mathbf{r}_e = \nabla\left[\frac{1}{2}|\mathbf{\Omega} \times \mathbf{r}|^2\right] . \tag{26.9}$$

According to (26.9), the centripetal acceleration can be regarded as the gradient of a potential, called the "centrifugal potential." The "apparent" gravity, $\mathbf{g}$, due to the combined effects of true gravity and the centripetal acceleration, can be defined as

$$\mathbf{g} = \mathbf{g}_a - \Omega^2 \mathbf{r}_e , \tag{26.10}$$

where $\mathbf{g}_a \equiv \nabla\Phi_a$, and using (26.9) we see that the potential of $\mathbf{g}$ is

$$\phi = \phi_a - \frac{1}{2}|\mathbf{\Omega} \times \mathbf{r}|^2 . \tag{26.11}$$

so that $\mathbf{g} \equiv \nabla\phi$. We refer to ϕ as the "geopotential." For most purposes $\mathbf{g} \cong \mathbf{g}_a = -g\mathbf{k}$, because the centripetal acceleration is small compared to $\mathbf{g}_a$. Here $\mathbf{k}$ is a unit vector pointing upward, away from the center of the Earth.

Using (26.11) we can now write the equation of motion (26.6) as

$$\frac{D\mathbf{V}}{Dt} = -2\mathbf{\Omega} \times \mathbf{V} - \nabla\phi - \alpha\nabla p - \alpha\nabla \bullet \mathbf{F} . \tag{26.12}$$

Another useful form of this equation is

$$\frac{\partial \mathbf{V}}{\partial t} + [2\mathbf{\Omega} + (\nabla \times \mathbf{V})] \times \mathbf{V} + \nabla(\tfrac{1}{2}\mathbf{V} \bullet \mathbf{V}) = -\nabla\phi - \alpha\nabla p - \alpha\nabla \bullet \mathbf{F} . \tag{26.13}$$

To obtain (26.13) from (26.12) we have used the vector identity

$$(\mathbf{V} \bullet \nabla)\mathbf{V} = (\nabla \times \mathbf{V}) \times \mathbf{V} + \nabla(\tfrac{1}{2}\mathbf{V} \bullet \mathbf{V}) . \tag{26.14}$$

We will have occasion to use both (26.12) and (26.13).

As an example of how the equation of motion can be expressed in a particular coordinate system, consider a *locally defined* "(x, y)" or Cartesian coordinate system, as shown in Fig. 26.2. Here "locally defined" means that the origin of the coordinate system is attached to a specific point on the Earth, e.g. Fort Collins. With this coordinate system,

$$\mathbf{\Omega} = (0, \Omega\cos\varphi, \Omega\sin\varphi) , \tag{26.15}$$

where φ is the latitude of the origin of the coordinate system, and

$$-\mathbf{\Omega} \times \mathbf{V} = -2\Omega \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & \cos\varphi & \sin\varphi \\ u & v & w \end{vmatrix} . \tag{26.16}$$

The directions of the unit vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are independent of position, although they do of course depend on the freely chosen position of the origin of the coordinate system.

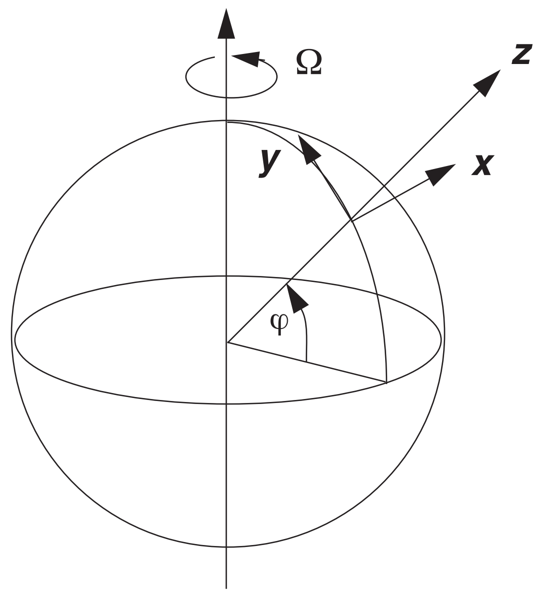

Figure 26.2: Sketch defining a local Cartesian coordinate system.

The components of the equation of motion in this local Cartesian coordinate system are

$$
\begin{aligned}
\frac{Du}{Dt} &= -\tilde{f}w + fv - \alpha\frac{\partial p}{\partial x} - \alpha(\nabla \bullet \mathbf{F})_x \,, \\
\frac{Dv}{Dt} &= -fu - \alpha\frac{\partial p}{\partial y} - \alpha(\nabla \bullet \mathbf{F})_y \,, \\
\frac{Dw}{Dt} &= \tilde{f}u - \alpha\frac{\partial p}{\partial z} - -\alpha(\nabla \bullet \mathbf{F})_z - g,
\end{aligned}
\tag{26.17}
$$

Here we define

$$\mathbf{V} \equiv u\mathbf{i} + v\mathbf{j} + w\mathbf{k} \,, \tag{26.18}$$

$$u \equiv \frac{Dx}{Dt} \,,\; v \equiv \frac{Dy}{Dt} \,,\; w \equiv \frac{Dz}{Dt} \,, \tag{26.19}$$

$$f \equiv 2\Omega\sin\varphi \,,\; f \equiv 2\Omega\cos\varphi \,, \tag{26.20}$$

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} \,. \tag{26.21}$$

At the origin, and at all points with the same longitude as the origin, the x-coordinate points east; but because this Cartesian coordinate system is defined with respect to a fixed location on the Earth's surface, the x-coordinate does not point east at other longitudes;

for example, at points 90° to the east of the origin the x-coordinate points up, and on the opposite side of the Earth from the origin the x-coordinate points west. Such a coordinate system *could* be used to study the general circulation, but clearly it is not very well suited to such an application.

As a possibly more familiar and certainly more useful possibility, consider spherical coordinates, (λ, φ, r). The unit vectors in the (λ, φ, r) coordinates are $\mathbf{e}_\lambda$, $\mathbf{e}_\varphi$, and $\mathbf{e}_r$, respectively.

You should be able to see that the direction of $\mathbf{e}_\lambda$ depends on longitude, and that the directions of $\mathbf{e}_\varphi$, and $\mathbf{e}_r$ depend on both longitude and latitude. This means that the directions of $\mathbf{e}_\lambda$, $\mathbf{e}_\varphi$, and $\mathbf{e}_r$ are functions of space, although of course their magnitudes are spatially constant. Simple geometrical reasoning leads to the following formulae:

$$\begin{aligned}\frac{D\mathbf{e}_\lambda}{Dt} &= \frac{D\lambda}{Dt}\sin\varphi\,\mathbf{e}_\varphi - \cos\varphi\frac{D\lambda}{Dt}\mathbf{e}_r \\ &= \left(\frac{u\tan\varphi}{a}\right)\mathbf{e}_\varphi - \frac{u}{a}\mathbf{e}_r \quad ,\end{aligned} \tag{26.22}$$

$$\begin{aligned}\frac{D\mathbf{e}_\varphi}{Dt} &= \frac{D\lambda}{Dt}\sin\varphi\,\mathbf{e}_\lambda - \frac{D\varphi}{Dt}\mathbf{e}_r \\ &= \left(\frac{u\tan\varphi}{a}\right)\mathbf{e}_\lambda - \frac{v}{a}\mathbf{e}_r \quad ,\end{aligned} \tag{26.23}$$

$$\begin{aligned}\frac{D\mathbf{e}_r}{Dt} &= \cos\varphi\frac{D\lambda}{Dt}\mathbf{e}_\lambda + \frac{D\varphi}{Dt}\mathbf{e}_\varphi \\ &= \frac{u}{a}\mathbf{e}_\lambda + \frac{v}{a}\mathbf{e}_\varphi \quad .\end{aligned} \tag{26.24}$$

The vector operators that will be used in this course can be expressed in spherical coordinates as follows:

$$\nabla A = \left(\frac{1}{r\cos\varphi}\frac{\partial A}{\partial\lambda}, \frac{1}{r}\frac{\partial A}{\partial\varphi}, \frac{\partial A}{\partial r}\right) , \tag{26.25}$$

$$\nabla \bullet \mathbf{V} = \frac{1}{r\cos\varphi}\frac{\partial V_\lambda}{\partial\lambda} + \frac{1}{r\cos\varphi}\frac{\partial}{\partial\varphi}(V_\varphi\cos\varphi) + \frac{1}{r^2}\frac{\partial}{\partial r}(V_r r^2) , \tag{26.26}$$

$$\nabla \times \mathbf{V} = \left\{ \frac{1}{r}\left[\frac{\partial V_r}{\partial \varphi} - \frac{\partial}{\partial r}(r V_\varphi) \right], \right.$$

$$\frac{1}{r}\frac{\partial}{\partial r}(r V_\lambda) - \frac{1}{r\cos\varphi}\frac{\partial V_r}{\partial \lambda},$$

$$\left. \frac{1}{r\cos\varphi}\left[\frac{\partial V_\varphi}{\partial \lambda} - \frac{\partial}{\partial \varphi}(V_\lambda \cos\varphi) \right] \right\}, \tag{26.27}$$

$$\nabla^2 A = \frac{1}{r^2\cos\varphi}\left[\frac{\partial}{\partial\lambda}\left(\frac{1}{\cos\varphi}\frac{\partial A}{\partial\lambda} \right) + \frac{\partial}{\partial\varphi}\left(\cos\varphi \frac{\partial A}{\partial\varphi} \right) + \frac{\partial}{\partial r}\left(r^2\cos\varphi \frac{\partial A}{\partial r} \right) \right]. \tag{26.28}$$

Here A is an arbitrary scalar, and $\mathbf{V}$ is an arbitrary vector.

Eq. (26.26) can be expanded as

$$\nabla \bullet \mathbf{V} = \frac{1}{r\cos\varphi}\frac{\partial V_\lambda}{\partial \lambda} + \frac{1}{r\cos\varphi}\frac{\partial}{\partial\varphi}(V_\varphi \cos\varphi) + \frac{\partial}{\partial r}V_r + \frac{2V_r}{r}. \tag{26.29}$$

Because the Earth's atmosphere is very thin compared to the radius of the Earth, the last term is negligible, and we can approximate the divergence operator by

$$\nabla \bullet \mathbf{V} = \frac{1}{a\cos\varphi}\left[\frac{\partial}{\partial\lambda}V_\lambda + \frac{\partial}{\partial\varphi}(V_\varphi\cos\varphi) \right] + \frac{\partial}{\partial r}V_r. \tag{26.30}$$

Note that r has been replaced by a in the first two terms. In this course, we normally use (26.30) rather than (26.26).

Define the velocity vector as

$$\mathbf{V} \equiv u\mathbf{e}_\lambda + v\mathbf{e}_\varphi + w\mathbf{e}_r, \tag{26.31}$$

where

$$\begin{aligned} u &\equiv r\cos\varphi\frac{D\lambda}{Dt},\ v \equiv r\frac{D\varphi}{Dt},\ w \equiv \frac{Dr}{Dt}, \\ \frac{D}{Dt} &\equiv \frac{\partial}{\partial t} + \frac{D\lambda}{Dt}\frac{\partial}{\partial\lambda} + \frac{D\varphi}{Dt}\frac{\partial}{\partial\varphi} + \frac{Dr}{Dt}\frac{\partial}{\partial r} \\ &= \frac{\partial}{\partial t} + \frac{u}{r\cos\varphi}\frac{\partial}{\partial\lambda} + \frac{v}{r}\frac{\partial}{\partial\varphi} + w\frac{\partial}{\partial r}. \end{aligned} \tag{26.32}$$

Note that in Eq. (26.32) the directions in which the u, v, and w components actually point depend on where you are. Then (26.13) can be written as

$$
\begin{aligned}
\frac{Du}{Dt}+\frac{uw}{r}-\frac{uv\tan\varphi}{r} &= fv-\hat{f}w-\frac{\alpha}{r\cos\varphi}\frac{\partial p}{\partial\lambda}-\alpha(\nabla\bullet\mathbf{F})_{\lambda}\,,\\
\frac{Dv}{Dt}+\frac{vw}{r}+\frac{u^2\tan\varphi}{r} &= -fu-\frac{\alpha}{r}\frac{\partial p}{\partial\varphi}-\alpha(\nabla\bullet\mathbf{F})_{\varphi}\,,\\
\frac{Dw}{Dt}-\left(\frac{u^2+v^2}{r}\right) &= \hat{f}u-\alpha\frac{\partial p}{\partial r}-\alpha(\nabla\bullet\mathbf{F})_r-g.
\end{aligned}
\tag{26.33}
$$

These are the components of the equation of motion in spherical coordinates. Compare with (26.17). Note that (26.17) does not contain the terms $\frac{uw}{r}$, $\frac{uv\tan\varphi}{r}$, etc. These so-called "metric" terms arise in (26.32) because the directions of the unit vectors $\mathbf{e}_{\lambda}$, $\mathbf{e}_{\varphi}$, and $\mathbf{e}_r$ vary with λ and φ.

By using the continuity equation in spherical coordinates, we can rewrite (26.33) in flux form:

$$
\begin{aligned}
\frac{\partial}{\partial t}(\rho u)+\nabla\bullet(\rho\mathbf{V}u)+\rho\frac{uw}{r}-\rho uv\frac{\tan\varphi}{r} &= \rho fv-\rho\hat{f}w-\frac{1}{r\cos\varphi}\frac{\partial p}{\partial\lambda}-(\nabla\bullet\mathbf{F})_{\lambda}\\
\frac{\partial}{\partial t}(\rho v)+\nabla\bullet(\rho\mathbf{V}v)+\rho\frac{vw}{r}+\rho u^2\frac{\tan\varphi}{r} &= -\rho fu-\frac{1}{r}\frac{\partial p}{\partial\varphi}-(\nabla\bullet\mathbf{F})_{\varphi}\,,\\
\frac{\partial}{\partial t}(\rho w)+\nabla\bullet(\rho\mathbf{V}w)-\rho\left(\frac{u^2+v^2}{r}\right) &= \rho\hat{f}u-\frac{\partial p}{\partial r}-\rho g-(\nabla\bullet\mathbf{F})_r\,.
\end{aligned}
\tag{26.34}
$$

These equations are fairly exact. Various approximations will be introduced later.

26.2 Solid-body rotation

It is instructive to consider the special case of solid-body rotation (SBR), in which

$$
u=\omega r\cos\varphi\,,\ v=0\,,\text{ and } w=0\,,\ p=p(\varphi,r)\,,
\tag{26.35}
$$

where $\omega = \text{constant}$. This type of motion is called "solid-body rotation" because the fluid rotates as if it were a solid, i.e., neighboring particles remain neighbors for all time. For SBR, Eqs. (26.34) reduce to

$$\frac{\partial}{\partial t}(\rho u) = 0 ,$$

$$\rho u^2 \frac{\tan\varphi}{r} = -\rho f u - \frac{1}{r}\frac{\partial p}{\partial \varphi} ,$$

$$-\rho\left(\frac{u^2}{r}\right) = \rho \hat{f} u - \frac{\partial p}{\partial r} - \rho g . \tag{26.36}$$

Here we have assumed that the stress tensor vanishes (it does). The zonal wind equation is in trivial balance. The meridional momentum equation is in "gradient-wind balance," which is a generalization of geostrophic balance. The vertical momentum equation is in a modified hydrostatic balance, in which the centripetal and coriolis accelerations enter.

26.3 Approximations

Up to here the discussion has been fairly exact. We now introduce some very useful approximations:

(i) Replace r by a everywhere, where a is the radius of the Earth. An approximation of this form can be justified for an atmosphere which is thin compared to the radius of the planet, and so it is called the "thin atmosphere approximation." It is a good approximation for Earth, but would not apply, e.g., to Jupiter.

(ii) Drop the terms containing $\hat{f}$. This means that the horizontal component of $\underset{\sim}{\Omega}$ disappears from the equations. This is often called "the traditional approximation." There is an ongoing discussion as to whether or not this is a good idea.

(iii) Neglect $\frac{uw}{r}$ and $\frac{vw}{r}$, the curvature terms involving w, in the equations for u and v, respectively, neglect $\frac{u^2 + v^2}{r}$ in the equation of vertical motion.

(iv) We now introduce a fourth, very familiar approximation, called the quasi-static approximation. For resting air, the vertical component of (26.34) reduces to

$$\frac{\partial p}{\partial z} = -\rho g . \tag{26.37}$$

This is called the hydrostatic equation. With an appropriate boundary condition, (26.37) allows us to compute $p(z)$ from $\rho(z)$. Even when the air is moving, (26.37) gives a good *approximation* to $p(z)$, simply because $\frac{Dw}{Dt}$ and the vertical component of the friction force are small compared to g. Eq. (26.37) as applied to moving air is called the hydrostatic approximation, and it is applicable to virtually all meteorological phenomena, including violent thunderstorms. For large-scale circulations, the approximate $p(z)$ determined through the use of (26.37) can be used to compute the pressure gradient force in the equation of horizontal motion. To do so is to use the *quasi-static approximation*. The quasi-static approximation applies very well for large-scale motions, but it is not applicable to many small-scale motions, such as thunderstorms. When the quasi-static approximation is made, the effective kinetic energy is due entirely to the horizontal motion; the contribution of the vertical component, w, is neglected. For large-scale motions, $w \ll (u, v)$, so that this quasistatic kinetic energy is very close to the true kinetic energy. Further discussion is given in a hand-out on the quasi-static approximation, available from the instructor.

With these four approximations, (26.33) is replaced by

$$\begin{aligned} \frac{Du}{Dt} - \frac{uv\tan\varphi}{a} &= fv - \frac{\alpha}{a\cos\varphi}\frac{\partial p}{\partial\lambda} - \alpha(\nabla \bullet \mathbf{F})_\lambda , \\ \frac{Dv}{Dt} + \frac{u^2\tan\varphi}{a} &= -fu - \frac{\alpha}{a}\frac{\partial p}{\partial\varphi} - \alpha(\nabla \bullet \mathbf{F})_\varphi , \\ 0 &= -g - \alpha\frac{\partial p}{\partial z} . \end{aligned} \tag{26.38}$$

26.4 Summary

We have presented the momentum equation that describes motion on a rotating sphere, as seen in the rotating frame of reference. We have also introduced several commonly used approximations.

References and Bibliography

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