The Moist Adiabatic Lapse Rate

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Basics of moist convection

In atmospheric science, the term "convection" refers to a buoyancy-driven circulation. In other fields this is often called "natural convection."

Before discussing moist convection, it is useful to briefly investigate dry convection. Consider the equation of vertical motion and the statement of approximate conservation of dry static energy, linearized with respect to a resting, horizontally uniform basic state:

$$\bar{\rho}\frac{\partial w'}{\partial t} = -\frac{\partial p'}{\partial z} - \rho' g \,, \tag{1}$$

$$\frac{\partial s'}{\partial t} = -w'\frac{\partial \bar{s}}{\partial z} \,. \tag{2}$$

Here the overbars denote horizontal averages, and primes denote departures from those averages. The gravity term of (1) represents the effects of buoyancy. Eq. (2) describes dry adiabatic motion.

We assume for simplicity that $\frac{\partial \bar{s}}{\partial z}$ is independent of height. Also for simplicity, we neglect the perturbation pressure term in (1), which generally acts to slightly reduce the buoyancy, and we use the approximation

$$-\left(\frac{\rho'}{\bar{\rho}}\right) \cong \frac{T'}{\bar{T}} = \frac{s'}{c_p\bar{T}} \,. \tag{3}$$

Then (1) reduces to

$$\frac{\partial w'}{\partial t} = \frac{gs'}{c_p\bar{T}} \,. \tag{4}$$

Eqs. (2) and (4) form a closed system. We look for solutions of the form

$$w'(t) = w'(0)\mathrm{Re}\{e^{\sigma t}\},$$
$$s'(t) = s'(0)\mathrm{Re}\{e^{\sigma t}\},$$

(5)

where σ may be either real or imaginary. Substituting, we find that for nontrivial solutions

$$\boxed{\sigma^2 = -\frac{g}{c_p\overline{T}}\frac{\partial\overline{s}}{\partial z}}.$$

(6)

For $\frac{\partial\overline{s}}{\partial z} < 0$, σ is real, and there is an exponentially growing solution with $\sigma > 0$; this is dry convective instability. We say that

$$\frac{\partial\overline{s}}{\partial z} < 0$$

(7)

is the criterion for dry convective instability. It can be seen from either (2) or (4) that in the exponentially growing solution, with $\sigma > 0$, $w'(t)$ and $s'(t)$ have the same sign for all time, so that $w's' > 0$, i.e., *convection transports dry static energy upward*. We will show in Chapter 7 that an upward temperature flux tends to lower the atmosphere's center of gravity, i.e., it reduces the total potential energy of the atmospheric column. The reduction in potential energy coincides with a generation of convective kinetic energy through the work done by the buoyancy force, so that the total energy is conserved. The generation of convective kinetic energy through an upward flux of dry static energy can be seen directly by multiplying both sides of (4) by w'.

For $\frac{\partial\overline{s}}{\partial z} > 0$, σ is imaginary, and the solutions are oscillatory; these are gravity waves. Their frequency, N, satisfies $N^2 = \frac{g}{c_p\overline{T}}\frac{\partial\overline{s}}{\partial z}$; this is called the Brunt-Väisällä frequency. Using the analysis given above as a starting point, you should be able to show that $\overline{w's'} = 0$ for a gravity wave, where the overbar represents an average over the period of the wave. This means that gravity waves do not transport dry static energy.

The analysis above shows that, in the absence of phase changes, convection and gravity waves are mutually exclusive; they cannot occur in the same place at the same time. We will see later that this conclusion does not necessarily apply when phase changes are allowed.

Up to this point, we have considered dry adiabatic motion. To analyze moist convection, we will assume saturated moist adiabatic motion. As discussed in Chapter 4, the moist static energy,

$$h \equiv c_p T + gz + Lq_v \, , \tag{8}$$

is approximately conserved under moist adiabatic processes. We replace (2) by

$$\frac{\partial h'}{\partial t} = -w' \frac{\partial \overline{h}}{\partial z} \, . \tag{9}$$

For saturated motion, the moist static energy must be equal to the saturation moist static energy, h^*, so we rewrite (6) as

$$\frac{\partial h^{*}{}'}{\partial t} = -w' \frac{\partial \overline{h^*}}{\partial z} \, . \tag{10}$$

where

$$h^* \equiv c_p T + gz + Lq^*\left(T, p\right), \tag{11}$$

and $q^*\left(T, p\right)$ is the saturation mixing ratio, which depends, as indicated, on temperature and pressure.

Next, we have to relate the buoyancy term of (6) to $h^{*}{}'$. Perturbations at fixed height, and at approximately fixed pressure, satisfy

$$h^{*}{}' \equiv c_p T'\left(1 + \gamma\right), \tag{12}$$

where, in the linearization,

$$\gamma \equiv \frac{L}{c_p}\left(\frac{\partial q^*}{\partial T}\right)_p \tag{13}$$

is evaluated using the mean-state temperature and pressure. The nondimensional parameter γ positive and of order one.

We now write

$$-\left(\frac{\rho'}{\bar{\rho}}\right) \cong \frac{T'}{\bar{T}} = \frac{h^{*}{}'}{c_p \bar{T}(1+\gamma)} . \tag{14}$$

Substitution into the equation of vertical motion gives

$$\frac{\partial w'}{\partial t} = \frac{g h^{*}{}'}{c_p \bar{T}(1+\gamma)} . \tag{15}$$

We look for exponential solutions of the system (10) and (15), and find that

$$\boxed{\sigma^2 = -\frac{g}{c_p \bar{T}(1+\gamma)} \frac{\partial \overline{h^*}}{\partial z}} . \tag{16}$$

This shows that the criterion for moist convective instability of a saturated atmosphere is

$$\frac{\partial \overline{h^*}}{\partial z} < 0 . \tag{17}$$

Compare with (6).

Before moving on, we need to do one more thing. The dry adiabatic lapse rate of temperature is given by

$$\begin{aligned} \Gamma_d &= -\left(\frac{\partial T}{\partial z}\right)_{\text{dry adiabatic}} \\ &= \frac{g}{c_p} . \end{aligned} \tag{18}$$

This is the rate at which temperature decreases with height when the dry static energy is independent of height. We can rewrite (2) as

$$\frac{\partial T'}{\partial t} = w'\left(\bar{\Gamma} - \Gamma_d\right) \tag{19}$$

This allows us to re-state the criterion for dry convective instability as

$$\Gamma > \Gamma_d . \tag{20}$$

We can also express the criterion for moist convective instability in terms of temperature, by starting from

$$\frac{\partial \overline{h^*}}{\partial z} = c_p \frac{\partial \overline{T}}{\partial z} + g + L\left[\left(\frac{\partial q^*}{\partial T}\right)_p \frac{\partial}{\partial z}\overline{T} + \left(\frac{\partial q^*}{\partial p}\right)_T \frac{\partial \overline{p}}{\partial z}\right]. \tag{21}$$

By using (13), (18), and the hydrostatic equation, this can be rearranged to

$$\frac{\partial \overline{h^*}}{\partial z} = c_p \left\{(1+\gamma)\frac{\partial \overline{T}}{\partial z} + \Gamma_d\left[1 - L\left(\frac{\partial q^*}{\partial p}\right)_T \overline{\rho}\right]\right\}. \tag{22}$$

For a saturated parcel rising moist adiabatically, $\frac{\partial \overline{h^*}}{\partial z} = 0$, so (22) leads to an expression for the moist adiabatic lapse rate, Γ_m, as

$$\begin{aligned}\Gamma_m &\equiv -\left(\frac{\partial \overline{T}}{\partial z}\right)_{\text{moist adiabatic}} \\ &= \frac{\Gamma_d}{(1+\gamma)}\left[1 - L\left(\frac{\partial q^*}{\partial p}\right)_T \overline{\rho}\right].\end{aligned} \tag{23}$$

To go further, we need formulas for $\left(\frac{\partial q^*}{\partial p}\right)_T$ and γ.

Recall that the saturation mixing ratio is defined by

$$q^*(T,p) \equiv \frac{\rho_v^*}{\rho_d}, \tag{24}$$

where ρ_v^* is the saturation vapor density, and ρ_d is the density of dry air. Both vapor and dry air obey the equation of state, with the same temperature but different gas constants:

$$p_d = \rho_d R_d T \text{ and } e = \rho_v R_v T. \tag{25}$$

Here e is the vapor pressure, and $R_v \cong 462 \text{ J kg}^{-1} \text{ K}^{-1}$. Taking the ratio of the two equations in (25), the temperatures cancel and we find that

$$\frac{e}{p_d} = \frac{\rho_v R_v}{\rho_d R_d}, \tag{26}$$

which is equivalent to

$$q \equiv \frac{\rho_v}{\rho_d} = \frac{\varepsilon e}{p_d}, \tag{27}$$

where

$$\varepsilon \equiv \frac{R_d}{R_v} \cong 0.622 . \tag{28}$$

Throughout the troposphere, $p_d \gg e$, so we can approximate (27) by

$$q \cong \frac{\varepsilon e}{p}, \tag{29}$$

and similarly

$$q^*(T,p) \cong \frac{\varepsilon e^*(T)}{p} . \tag{30}$$

A simple but approximate formula for $e^*(T)$ is

$$e^*(T) \cong 6.11 \exp\left[\frac{L}{R_v}\left(\frac{1}{273} - \frac{1}{T}\right)\right] \text{ hPa.} \tag{31}$$

From (30), we see that

$$\left(\frac{\partial q^*}{\partial p}\right)_T \cong -\frac{q^*(T,p)}{p} < 0 , \tag{32}$$

and

$$\gamma \equiv \frac{L}{c_p}\left(\frac{\partial q^*}{\partial T}\right)_p = \frac{L}{c_p}\frac{\varepsilon}{p}\frac{d}{dT}e^*(T) . \tag{33}$$

The Clausius-Clapeyron equation can be closely approximated by

$$\frac{d}{dT}e^*(T) \cong \frac{Le^*(T)}{R_v T^2}\,. \tag{34}$$

Substitution into (32) gives

$$\begin{aligned}\gamma &\equiv \frac{L}{c_p}\left(\frac{\partial q^*}{\partial T}\right)_p \\ &= \frac{L^2 q^*(T,p)}{c_p R_v T^2}\,.\end{aligned} \tag{35}$$

Then (23) can be written as

$$\boxed{\Gamma_m = \Gamma_d \left[\frac{1+\dfrac{Lq^*(T,p)}{R_d T}}{1+\dfrac{L^2 q^*(T,p)}{c_p R_v T^2}}\right]}\,. \tag{36}$$

The denominator of (36) is larger than the numerator, so $\Gamma_m < \Gamma_d$, although $\Gamma_m \to \Gamma_d$ at cold temperatures. For example, with a pressure of 1000 mb and a temperature of 288 K, we find that

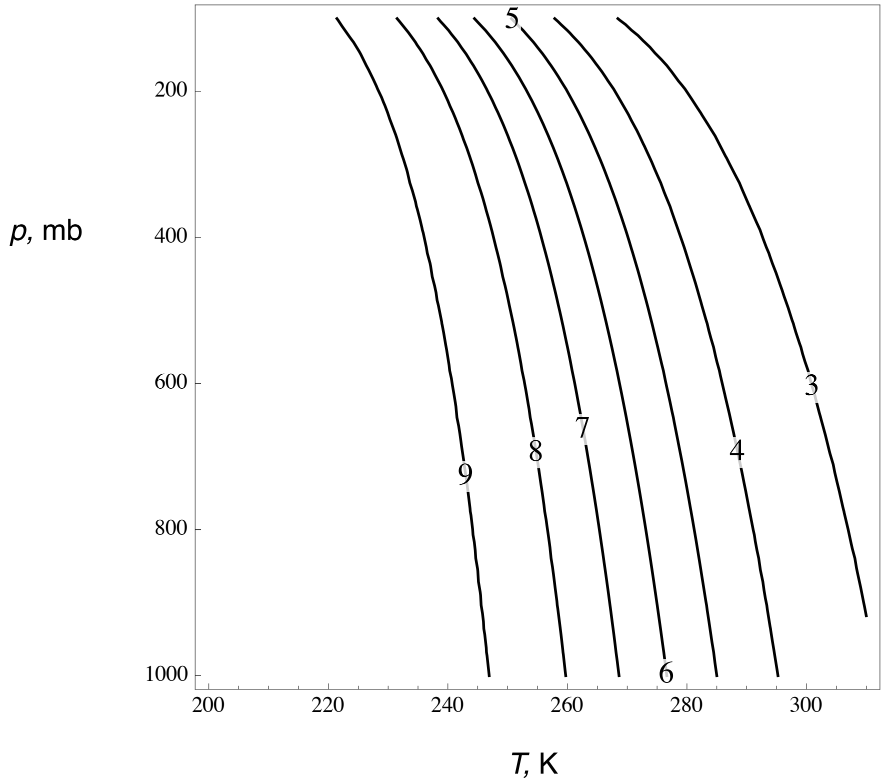

Figure 1: A plot of the moist adiabatic lapse rate, Γ_m, in K km^{-1}, as a function of temperature and pressure.

$\Gamma_m = 4.67$ K km^{-1} (see Fig. 1). As the temperature increases, the moist adiabatic lapse rate decreases.

For saturated motion, we can rewrite (2) as

$$\frac{\partial T'}{\partial t} = w'\left(\bar{\Gamma} - \Gamma_m\right) .$$

(37)

(38)