

Hermite Polynomials

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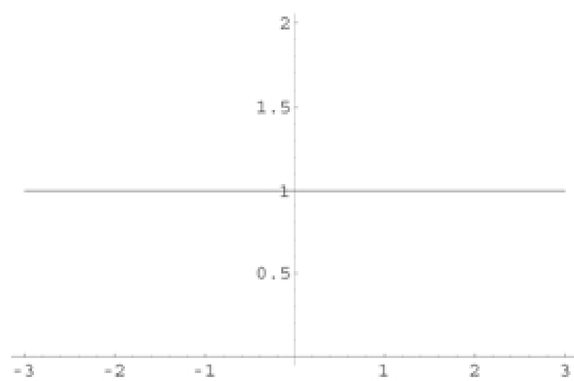
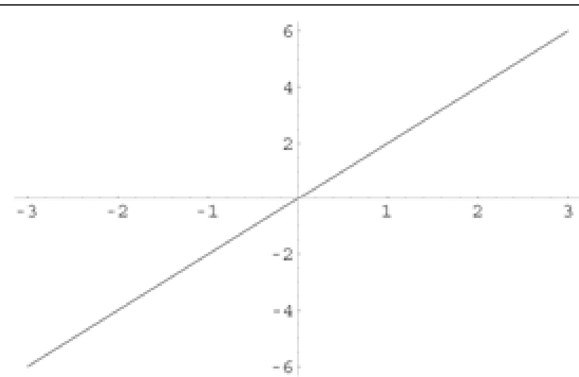
The Hermite polynomials are defined by:

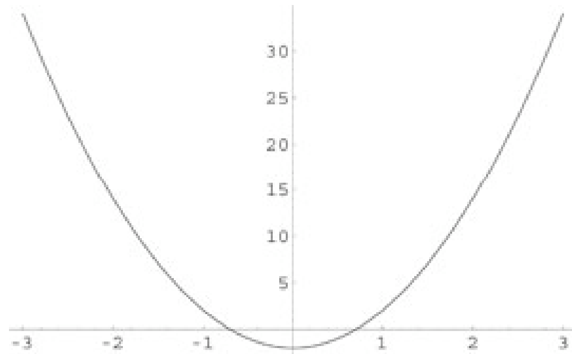
$$H_n(y) \equiv (-1)^n e^{y^2} \frac{d^n}{d y^n} \left(e^{-y^2}\right), \text{ for } n \geq 0. \tag{1}$$

The first six Hermite polynomials are given and plotted in Table 1.

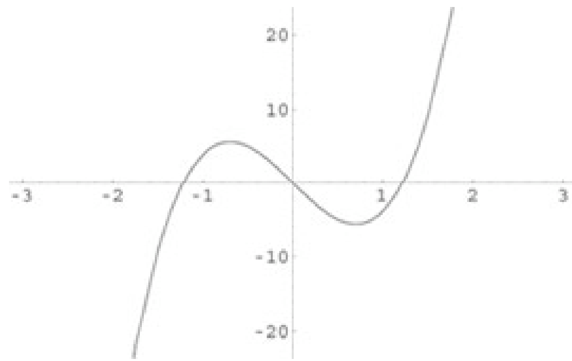
Table 1: Examples of Hermite Polynomials.

Hermite Polynomial	*Plot*
$H_0(y) = 1$	
$H_1(y) = 2y$	

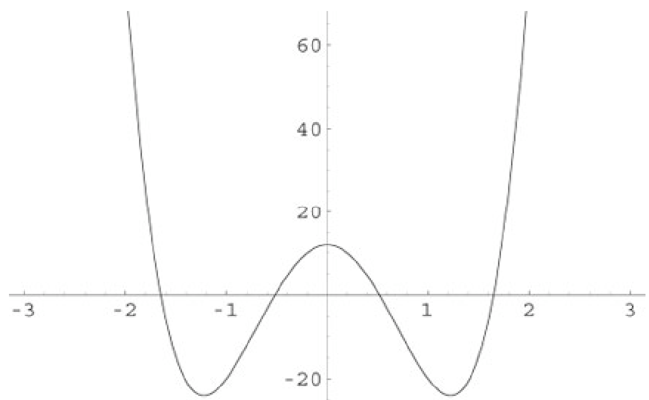
$H_2(y) = 4\,y^2 - 2$



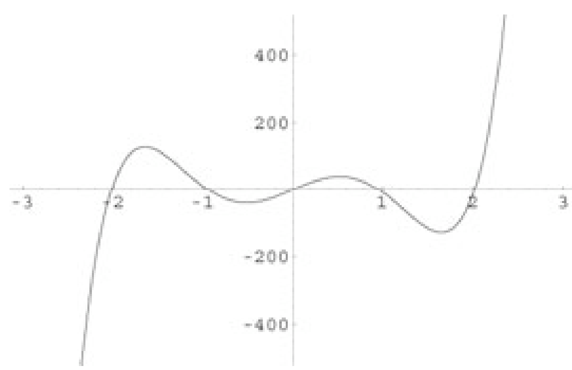
$H_3(y) = 8\,y^3 - 12\,y$



$H_4(y) = 16\,y^4 - 48\,y^2 + 12$



$H_5(y) = 32\,y^5 - 160\,y^3 + 120\,y$



Note that the even-numbered Hermite polynomials are even functions, and the odd-numbered Hermite polynomials are odd functions.

The Hermite polynomials are orthogonal with respect to e^{-y^2} :

$$\int_{-\infty}^{\infty} H_n(y)\, H_m(y)\, e^{-y^2}\, dy = 0 \text{ for } n \neq m \ . \tag{2}$$

$$\int_{-\infty}^{\infty} H_n(y)\, H_n(y)\, e^{-y^2}\, dy = 2^n\, n! \text{ for } n\sqrt{\pi} \ . \tag{3}$$

A useful recursion formula is

$$\frac{d}{dy} H_n(y) = 2\, n H_{n-1}(y) \text{ for } n \geq 1 \ . \tag{4}$$

Let

$$\psi_n(y) \equiv \frac{e^{-y^2/2}\, H_n(y)}{\sqrt{2^n\, n!\, \pi^{1/2}}} \ . \tag{5}$$

The $\psi_n(y)$ satisfy

$$\left\{ \frac{d^2}{dy^2} + \left[(2\,n+1) - y^2 \right] \right\} \psi_n = 0 \ , \tag{6}$$

as can be verified by substitution. Note that for $n \geq 0$, the expression $2\,n+1$ generates all positive odd integers.

References & Bibliography

Courant, R., and D. Hilbert, 1989: *Methods of Mathematical Physics, Volume 1*. Wiley Interscience, 560 pp.