Hermite Polynomials

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The Hermite polynomials are defined by:

\[ H_n(y) \equiv (-1)^n e^{y^2} \frac{d^n}{dy^n} \left( e^{-y^2} \right), \text{ for } n \geq 0. \]

(1)

The first six Hermite polynomials are given in the Table below:

<table>
<thead>
<tr>
<th>y</th>
<th>0.5</th>
<th>1.0</th>
<th>1.5</th>
<th>2.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>-3</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>-2</td>
<td></td>
<td></td>
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<td></td>
</tr>
<tr>
<td>-1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>1.0</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
<td>1.0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
<td>1.0</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td>1.0</td>
</tr>
</tbody>
</table>
\[ H_1(y) = 2y \]

\[ H_2(y) = 4y^2 - 2 \]

\[ H_3(y) = 8y^3 - 12y \]
Note that the even-numbered Hermite polynomials are even functions, and the odd-numbered Hermite polynomials are odd functions.

The Hermite polynomials are orthogonal with respect to $e^{-y^2}$:

$$\int_{-\infty}^{\infty} H_n(y) H_m(y) e^{-y^2} \, dy = 0 \quad \text{for} \quad n \neq m,$$

(2)

$$\int_{-\infty}^{\infty} H_n(y) H_m(y) e^{-y^2} \, dy = 2^n n! \sqrt{\pi} \quad \text{for} \quad n = m.$$

(3)

Two useful recursion relations are
\[ \frac{d}{dy} H_n(y) = 2nH_{n-1}(y) \quad \text{for } n \geq 1. \]  

(4)

and

\[ H_{n+1}(y) = 2yH_n(y) - 2nH_{n-1}(y). \]  

(5)

Let

\[ \psi_n(y) \equiv \frac{e^{-y^2/2} H_n(y)}{\sqrt{2^n n! \pi^{1/2}}}. \]  

(6)

The \( \psi_n(y) \) satisfy

\[ \left\{ \frac{d^2}{dy^2} + \left( 2n + 1 - y^2 \right) \right\} \psi_n = 0, \]  

(7)

as can be verified by substitution. Note that for \( n \geq 0 \), the expression \( 2n + 1 \), which appears in (7), generates all positive odd integers.

**References and Bibliography**