Suppose that we have a superposition of two waves, with wave numbers $k_1$ and $k_2$, respectively. Then we can write

\[
\exp[ik_1(x - c_1 t)] + \exp[ik_2(x - c_2 t)] \\
\approx \exp\{i(k - \Delta k)x - (kc + \Delta(kc))t\} + \exp\{i(k + \Delta k)x - (kc + \Delta(kc))t\} \\
= \exp[ik(x - ct)](\exp\{i[\Delta kx - \Delta(kc)]t\} + \exp\{-i[\Delta kx - \Delta(kc)]t\}) \\
= 2 \cos[\Delta kx - \Delta(kc)t]\exp\{ik(x - ct)\}.
\]

Now define

\[
k \equiv \frac{k_1 + k_2}{2}, \quad c \equiv \frac{c_1 + c_2}{2}, \quad \Delta k \equiv \frac{k_1 - k_2}{2}, \quad \Delta(kc) \equiv \frac{k_1 c_1 + k_2 c_2}{2}.
\]

Note that $k_1 = k + \Delta k$ and $k_2 = k - \Delta k$ In the first line of (1), the exponents are the same within $\frac{1}{2} \Delta k$. If $\Delta k$ is very small, the factor $\cos\Delta k[x - \frac{\Delta(kc)}{\Delta k}t]$ may appear schematically as the outer, slowly varying curves in the figure below. It “modulates” wave $k$, which is represented by the inner, rapidly varying curve in the figure. The wavelets (dashed lines) move with phase speed $c = \frac{kc}{k}$, but the wave packets (the solid curves, forming an envelope of the wavelets) move with speed $\frac{\Delta(kc)}{\Delta k}$. The differential expression $\frac{d(kc)}{dk} = c_g$ is called the “group velocity.” Note that $c_g = c$ if $c$ does not depend on $k$. 

"Group Velocity"

David A. Randall

Department of Atmospheric Science
Colorado State University, Fort Collins, Colorado 80523
Figure 1: Sketch used to illustrate the concept of group velocity. The short waves are modulated by longer waves.