Assume that \( u(x,t) \) is real and integrable. If the domain is periodic, with period \( L \), we can express \( u(x,t) \) exactly by a Fourier series expansion:

\[
u(x,t) = \sum_{k=-\infty}^{\infty} \hat{u}_k(t) e^{ikx} .
\]

The complex coefficients \( \hat{u}_k(t) \) can be evaluated using

\[
\hat{u}_k(t) = \frac{1}{L} \int_{x-L/2}^{x+L/2} u(x,t) e^{-ikx} \, dx.
\]

Recall that the proof of (1) and (2) involves use of the orthogonality condition

\[
\frac{1}{L} \int_{x-L/2}^{x+L/2} e^{ikx} e^{ilx} \, dx = \delta_{kl} ,
\]

where

\[
\delta_{kl} \equiv \begin{cases} 1, & k = l \\ 0, & k \neq l \end{cases}
\]

is the Kronecker delta.

From (1), we see that the \( x \)-derivative of \( u \) satisfies

\[
\frac{\partial u}{\partial x}(x,t) = \sum_{k=-\infty}^{\infty} ik\hat{u}_k(t) e^{ikx} .
\]
Inspection of (5) shows that \( \frac{\partial u}{\partial x} \) does not have a contribution from \( \hat{u}_0 \); the reason for this should be clear.

A numerical model uses equations similar to (1), (2), and (5), but with a finite set of wave numbers, and with \( x \) defined on a finite mesh:

\[
u(x_j, t) \equiv \sum_{k=-n}^{n} \hat{u}_k(t) e^{ikx_j},
\]

(6)

\[
\hat{u}_k(t) \equiv \frac{1}{M} \sum_{l=1}^{M} u(x_j, t) e^{-ilkx_j}, -n \leq k \leq n,
\]

(7)

\[
\frac{\partial u}{\partial x}(x_j, t) \equiv \sum_{k=-n}^{n} i k \hat{u}_k(t) e^{ikx_j}.
\]

(8)

Note that we have used “approximately equal signs” in (6) - (8). In (7) we sum over a grid with \( M \) points. In the following discussion, we assume that the value of \( n \) is chosen by the user. The value of \( M \), corresponding to a given value of \( n \), is discussed below.

Substitution of (6) into (7) gives

\[
\hat{u}_k(t) = \frac{1}{M} \sum_{l=1}^{M} \left[ \sum_{j=-n}^{n} \hat{u}_l(t) e^{i k x_j} \right] e^{-i k x_j}, -n \leq k \leq n.
\]

(9)

This is of course a rather circular substitution, but the result serves to clarify some basic ideas. If expanded, each term on the right-hand side of (9) involves the product of two wave numbers, \( l \) and \( k \), each of which lies in the range \(-n\) to \( n\). The range for wave number \( l \) is explicitly spelled out in the inner sum on the right-hand side of (9); the range for wave number \( k \) is understood because, as indicated, we wish to evaluate the left-hand side of (9) for \( k \) in the range \(-n\) to \( n\). Because each term on the right-hand side of (9) involves the product of two Fourier modes with wave numbers in the range \(-n\) to \( n\), each term includes wave numbers up to \( \pm 2n\). We therefore need \( 2n+1 \) complex coefficients, i.e. \( 2n+1 \) values of the \( \hat{u}_k(t) \).

Because \( u \) is real, it must be true that \( \hat{u}_{-k} = \hat{u}_k^* \), where the * denotes the conjugate. To see why this is so, consider the \( +k \) and \( -k \) contributions to the sum in (6):
\[ T_k(x_j) = \hat{u}_k(t)e^{ikx_j} + \hat{u}_{-k}(t)e^{-ikx_j} = R_k e^{i\theta} e^{ikx_j} + R_{-k} e^{i\mu} e^{-ikx_j}, \]  
\text{(10)}

where \( R_k e^{i\theta} = \hat{u}_k(t) \) and \( R_{-k} e^{i\mu} = \hat{u}_{-k}(t) \), and \( R_k \) and \( R_{-k} \) are real and non-negative. By linear independence, our assumption that \( u(x_j,t) \) for all \( x_j \) is real implies that the imaginary part of \( T_k(x_j) \) must be zero, for all \( x_j \). It follows that

\[ R_k \sin(\theta + kx_j) + R_{-k} \sin(\mu - kx_j) = 0 \text{ for all } x_j. \]  
\text{(11)}

The only way to satisfy this for all \( x_j \) is to set

\[ \theta + kx_j = -(\mu - kx_j) = -\mu + kx_j, \]  
which means that \( \theta = -\mu, \)  
\text{(12)}

and

\[ R_k = R_{-k}. \]  
\text{(13)}

Eqs. (12) and (13) imply that

\[ \hat{u}_{-k} = \hat{u}_k^*, \]  
\text{(14)}

as was to be demonstrated.

Eq. (14) implies that \( \hat{u}_k \) and \( \hat{u}_{-k} \) together involve only two distinct real numbers. In addition, it follows from (14) that \( \hat{u}_0 \) is real. Therefore, the \( 2n + 1 \) complex values of \( \hat{u}_k \) embody the equivalent of only \( 2n + 1 \) distinct real numbers. The Fourier representation up to wave number \( n \) is thus equivalent to representing the real function \( u(x,t) \) on \( 2n + 1 \) grid points, in the sense that the information content is the same. We conclude that, in order to use a grid of \( M \) points to represent the amplitudes and phases of all waves up to \( k = \pm n \), we need \( M \geq 2n + 1 \); we can use more than \( 2n + 1 \) points, but not fewer.

As a very simple example, a highly truncated Fourier representation of \( u \) including just wave numbers zero and one is equivalent to a grid-point representation of \( u \) using 3 grid points. The real values of \( u \) assigned at the three grid points suffice to compute the coefficient of wave number zero (the mean value of \( u \)) and the phase and amplitude (or “sine and cosine coefficients”) of wave number one.

Substituting (7) into (8) gives
\[ \frac{\partial u}{\partial x}(x, t) \equiv \sum_{k=-n}^{n} \left[ \frac{ik}{M} \sum_{j=1}^{M} u(x_j, t) e^{-ikx_j} \right] e^{ikx}. \]  

(15)

Reversing the order of summation leads to

\[ \frac{\partial u}{\partial x}(x, t) \equiv \sum_{j=1}^{M} \alpha_j u(x_j, t), \]  

(16)

where

\[ \alpha_j \equiv \frac{i}{M} \sum_{k=-n}^{n} ke^{ik(x_j-x_l)}. \]  

(17)

The point of this little derivation is that (16) can be interpreted as a finite-difference approximation - a special one involving all grid points in the domain. From this point of view, spectral models can be regarded as a class of finite-difference models.

Eq. (9) can be rewritten as

\[ (\hat{u}_k) = \frac{1}{M} \sum_{j=1}^{M} \left[ \sum_{l=-n}^{n} \hat{u}_l e^{i(l-k)x_j} \right]. \]  

(18)

The contribution of wave number \( l \) to \( \hat{u}_k \) is

\[ (\hat{u}_k)_l = \frac{1}{M} \sum_{j=1}^{M} \left[ \hat{u}_l e^{i(l-k)x_j} \right] = \frac{\hat{u}_l}{M} \sum_{j=1}^{M} e^{i(l-k)x_j}. \]  

(19)

Then we can write

\[ \hat{u}_k = \sum_{l=-n}^{n} (\hat{u}_k)_l. \]  

(20)

For \( l = k \), we recover

\[ (\hat{u}_k)_k = \frac{\hat{u}_k}{M} \sum_{j=1}^{M} 1 = \hat{u}_k. \]  

(21)

For \( l \neq k \), we get
\[
\left( \hat{u}_k \right)_l = \frac{\hat{u}_l}{M} \sum_{i=1}^{M} \left\{ \cos \left[ (l - k) x_j \right] + i \sin \left[ (l - k) x_j \right] \right\}.
\] (22)

From (22) we can see that, provided that \( l - k \) “fits” on the grid, we will have

\[
\left( \hat{u}_k \right)_l = 0 \text{ for } l \neq k.
\] (23)

The results (21) and (23) are as would be expected.

The contribution of grid point \( j \) to the Fourier coefficient \( \hat{u}_k \) is

\[
\left( \hat{u}_k \right)_j = \frac{1}{M} \sum_{l=-n}^{n} \hat{u}_l e^{i(l-k)x_j}.
\] (24)

Then

\[
\hat{u}_k = \sum_{l=1}^{M} \left( \hat{u}_k \right)_j, \quad -n \leq k \leq n
\] (25)

We must get \( 2n + 1 \) independent complex values of \( \hat{u}_k \). These are

\[
\left( \hat{u}_k \right)_j = \frac{e^{-ikx_j}}{M} \sum_{l=-n}^{n} \hat{u}_l e^{ibx_j}.
\] (26)

Comparing with (7), we see that

\[
\left( \hat{u}_k \right)_j = u(x_j) e^{ikx_j}, \quad -n \leq k \leq n,
\] (27)

as expected.