

A derivation of the governing equations for the eddy kinetic energy, zonal kinetic energy, and total kinetic energy

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What follows is a fairly detailed derivation of the governing equations for eddy kinetic energy, zonal kinetic energy, and total kinetic energy. Students should work through these details because only by doing so is it possible to understand where the equations come from, and to become comfortable with them. There are similar derivations for many other eddy statistics, and but those are omitted here for brevity.

The "flux forms" of the primitive forms of the zonal and meridional momentum equations, in spherical coordinates, with pressure as the vertical coordinate, are given by:

$$\begin{aligned}\frac{\partial u}{\partial t} + \frac{1}{a\cos\varphi}\frac{\partial}{\partial\lambda}(uu) + \frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}(vu\cos\varphi) + \frac{\partial}{\partial p}(\omega u) \\ = -\frac{1}{a\cos\varphi}\frac{\partial\phi}{\partial\lambda} + \left(f + u\frac{\tan\varphi}{a}\right)v + g\frac{\partial\tau_\lambda}{\partial p},\end{aligned} \tag{2.1}$$

$$\begin{aligned}\frac{\partial v}{\partial t} + \frac{1}{a\cos\varphi}\frac{\partial}{\partial\lambda}(uv) + \frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}(vv\cos\varphi) + \frac{\partial}{\partial p}(\omega v) \\ = -\frac{1}{a}\frac{\partial\phi}{\partial\varphi} - \left(f + u\frac{\tan\varphi}{a}\right)u + g\frac{\partial\tau_\varphi}{\partial p}.\end{aligned} \tag{2.2}$$

The corresponding continuity equation is:

$$\frac{1}{a\cos\varphi}\frac{\partial u}{\partial\lambda} + \frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}(v\cos\varphi) + \frac{\partial\omega}{\partial p} = 0. \tag{2.3}$$

Finally, the hydrostatic equation is:

$$\frac{\partial\phi}{\partial p} = -\alpha. \tag{2.4}$$

Now we introduce our notation for zonal means and eddies, and write, for example,

$$u = [u] + u^*. \tag{2.5}$$

Then we obtain the following zonally averaged forms of (2.1)-(2.4):

$$\frac{\partial[u]}{\partial t}+\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}\{[v][u]\cos\varphi+[v^{*}u^{*}]\cos\varphi\}+\frac{\partial}{\partial p}\{[\omega][u]+[\omega^{*}u^{*}]\}$$
$$=([v][u]+[v^{*}u^{*}])\frac{\tan\varphi}{a}+f[v]+g\frac{\partial}{\partial p}[\tau_{\lambda}], \tag{2.6}$$

$$\frac{\partial[v]}{\partial t}+\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}\{[v][v]\cos\varphi+[v^{*}v^{*}]\cos\varphi\}+\frac{\partial}{\partial p}(\{[\omega][v]+[\omega^{*}v^{*}]\})$$
$$=-([u][u]+[u^{*}u^{*}])\frac{\tan\varphi}{a}-f[u]-\frac{1}{a}\frac{\partial[\phi]}{\partial\varphi}+g\frac{\partial}{\partial p}[\tau_{\varphi}], \tag{2.7}$$

$$\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}\{[v]\cos\varphi\}+\frac{\partial}{\partial p}[\omega]=0, \tag{2.8}$$

$$\frac{\partial}{\partial p}[\phi]=-[\alpha]. \tag{2.9}$$

By subtracting (2.6) from (2.1), we obtain the zonal eddy momentum equation

$$\frac{\partial u^*}{\partial t}+\frac{1}{a\cos\varphi}\frac{\partial}{\partial\lambda}(2u^*[u]+u^*u^*)$$
$$+\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}\{([v]u^*+v^*[u]+v^*u^*-[v^*u^*])\cos\varphi\}$$
$$+\frac{\partial}{\partial p}([\omega]u^*+\omega^*[u]+\omega^*u^*-[\omega^*u^*]) \tag{2.10}$$
$$=-\frac{1}{a\cos\varphi}\frac{\partial}{\partial\lambda}(\phi^*)+\{u^*[v]+[u]v^*+v^{*}u^{*}-[v^{*}u^{*}]\}\frac{\tan\varphi}{a}+fv^*+g\frac{\partial}{\partial p}\tau_{\lambda}^*.$$

This is a flux form. We can convert it to an advective form by using the zonally averaged form of the continuity equation, i.e. (2.8), and also the eddy form of the continuity equation, which is:

$$\frac{1}{a\cos\varphi}\frac{\partial}{\partial\lambda}u^*+\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}(v^*\cos\varphi)+\frac{\partial}{\partial p}\omega^*=0. \tag{2.11}$$

The resulting advective form of the zonal eddy momentum equation is:

$$
\begin{aligned}
&\frac{\partial u^*}{\partial t} + \frac{[u]}{a\cos\varphi}\frac{\partial}{\partial\lambda}u^* + \frac{u^*}{a\cos\varphi}\frac{\partial}{\partial\lambda}u^* \\
&+ \frac{[v]}{a}\frac{\partial}{\partial\varphi}u^* + \frac{v^*}{a}\frac{\partial}{\partial\varphi}([u]+u^*) - \frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}([v^*u^*]\cos\varphi) \\
&+ [\omega]\frac{\partial}{\partial p}u^* + \omega^*\frac{\partial}{\partial p}([u]+u^*) - \frac{\partial}{\partial p}[\omega^*u^*] \\
&= -\frac{1}{a\cos\varphi}\frac{\partial}{\partial\lambda}(\phi^*) + \{u^*[v]+[u]v^*+v^*u^*-[v^*u^*]\}\frac{\tan\varphi}{a} + fv^* + g\frac{\partial}{\partial p}\tau_\lambda^*.
\end{aligned}
\tag{2.12}
$$

The next step is to multiply (2.12) by u^*:

$$
\begin{aligned}
&\frac{\partial}{\partial t}\frac{(u^*)^2}{2} + \frac{[u]}{a\cos\varphi}\frac{\partial}{\partial\lambda}\frac{(u^*)^2}{2} + \frac{u^*}{a\cos\varphi}\frac{\partial}{\partial\lambda}\frac{(u^*)^2}{2} \\
&+ \frac{[v]}{a}\frac{\partial}{\partial\varphi}\frac{(u^*)^2}{2} + \frac{v^*u^*}{a}\frac{\partial}{\partial\varphi}[u] + \frac{v^*}{a}\frac{\partial}{\partial\varphi}\frac{(u^*)^2}{2} - \frac{u^*}{a\cos\varphi}\frac{\partial}{\partial\varphi}([v^*u^*]\cos\varphi) \\
&+ [\omega]\frac{\partial}{\partial p}\frac{(u^*)^2}{2} + \omega^*u^*\frac{\partial}{\partial p}[u] + \omega^*\frac{\partial}{\partial p}\frac{(u^*)^2}{2} - u^*\frac{\partial}{\partial p}([\omega^*u^*]) \\
&= -\frac{u^*}{a\cos\varphi}\frac{\partial}{\partial\lambda}(\phi^*) + u^*\{u^*[v]+[u]v^*+v^*u^*-[v^*u^*]\}\frac{\tan\varphi}{a} + fv^*u^* + u^*g\frac{\partial}{\partial p}\tau_\lambda^*.
\end{aligned}
\tag{2.13}
$$

Now we reverse our steps, using the zonal mean and eddy forms of the continuity equation to go back into flux form:

$$
\begin{aligned}
&\frac{\partial}{\partial t}\frac{(u^*)^2}{2} + \frac{1}{a\cos\varphi}\frac{\partial}{\partial\lambda}\frac{(u^*)^3}{2} \\
&+ \frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}\frac{\{[v](u^*)^2\cos\varphi\}}{2} + \frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}\frac{\{v^*(u^*)^2\cos\varphi\}}{2} \\
&+ \frac{\partial}{\partial p}\left\{[\omega]\frac{(u^*)^2}{2}\right\} + \frac{\partial}{\partial p}\left\{\omega^*\frac{(u^*)^2}{2}\right\} \\
&+ v^*u^*\frac{\partial}{\partial\varphi}[u] + \omega^*u^*\frac{\partial}{\partial p}[u] - \frac{u^*}{a\cos\varphi}\frac{\partial}{\partial\varphi}([v^*u^*]\cos\varphi) - u^*\frac{\partial}{\partial p}([\omega^*u^*]) \\
&= -\frac{u^*}{a\cos\varphi}\frac{\partial}{\partial\lambda}(\phi^*) + u^*\{u^*[v]+[u]v^*+v^*u^*-[v^*u^*]\}\frac{\tan\varphi}{a} + fv^*u^* + u^*g\frac{\partial}{\partial p}\tau_\lambda^*.
\end{aligned}
\tag{2.14}
$$

Next, this equation is zonally averaged, resulting in considerable simplification, and yielding

$$
\begin{aligned}
&\frac{\partial}{\partial t}\left[\frac{(u^*)^2}{2}\right]+\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}\left\{\left([v]\left[\frac{(u^*)^2}{2}\right]+\frac{1}{2}[v^*u^*u^*]\right)\cos\varphi\right\} \\
&+\frac{\partial}{\partial p}\left\{[\omega]\left[\frac{(u^*)^2}{2}\right]+\frac{1}{2}[\omega^*u^*u^*]\right\} \\
&=-\frac{[v^*u^*]}{a}\frac{\partial}{\partial\varphi}[u]-[\omega^*u^*]\frac{\partial}{\partial p}[u] \\
&-\left[\frac{u^*}{a\cos\varphi}\frac{\partial}{\partial\lambda}(\phi^*)\right]+\{[u^*u^*][v]+[u][v^*u^*]+[v^*u^*u^*]\}\frac{\tan\varphi}{a}+f[v^*u^*] \\
&+\left[u^*g\frac{\partial}{\partial p}\tau_\lambda{}^*\right].
\end{aligned}
\tag{2.15}
$$

We now follow essentially the same logic with the meridional momentum equation. By subtracting (2.7) from (2.2), we obtain an equation governing the meridional eddy momentum:

$$
\begin{aligned}
&\frac{\partial v^*}{\partial t}+\frac{1}{a\cos\varphi}\frac{\partial}{\partial\lambda}(u^*[v]+[u]v^*+u^*v^*) \\
&+\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}\{(2[v]v^*+v^*v^*-[v^*v^*])\cos\varphi\} \\
&+\frac{\partial}{\partial p}([\omega]v^*+\omega^*[v]+\omega^*v^*-[\omega^*v^*]) \\
&=-\frac{1}{a}\frac{\partial}{\partial\varphi}(\phi^*)-\{2u^*[u]+u^*u^*-[u^*u^*]\}\frac{\tan\varphi}{a}-fu^*+g\frac{\partial}{\partial p}\tau_\varphi{}^*
\end{aligned}
\tag{2.16}
$$

Next, convert to advective form, multiply by v^*, convert back to flux form, and zonally average, to obtain

$$\frac{\partial}{\partial t}\left[\frac{(v^*)^2}{2}\right] + \frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}\left\{\left([v]\left[\frac{(v^*)^2}{2}\right] + \frac{1}{2}[v^*v^*v^*]\right)\cos\varphi\right\}$$
$$+\frac{\partial}{\partial p}\left([\omega]\left[\frac{(v^*)^2}{2}\right] + \frac{1}{2}[\omega^*v^*v^*]\right)$$
$$= -\frac{[v^*v^*]}{a}\frac{\partial}{\partial\varphi}[v] - [\omega^*v^*]\frac{\partial}{\partial p}[v] \tag{2.17}$$
$$-\left[\frac{v^*}{a}\frac{\partial}{\partial\varphi}(\phi^*)\right] - \{2[u^*v^*][u] + [u^*u^*v^*]\}\frac{\tan\varphi}{a} - f[u^*v^*]$$
$$+\left[v^*g\frac{\partial}{\partial p}\tau_\varphi{}^*\right].$$

This is analogous to (2.15).

We can now add (2.15) and (2.17), and define

$$KE \equiv \frac{1}{2}[(u^*)^2 + (v^*)^2], \tag{2.18}$$

to obtain the *eddy kinetic energy equation*:

$$\frac{\partial}{\partial t}KE + \frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}\left\{\left([v]KE + \frac{1}{2}[v^*u^*u^*] + \frac{1}{2}[v^*v^*v^*]\right)\cos\varphi\right\}$$
$$+\frac{\partial}{\partial p}\left([\omega]KE + \frac{1}{2}[\omega^*u^*u^*] + \frac{1}{2}[\omega^*v^*v^*]\right)$$
$$= \frac{-[v^*u^*]}{a}\frac{\partial}{\partial\varphi}[u] - \frac{[v^*v^*]}{a}\frac{\partial}{\partial\varphi}[v] - [\omega^*u^*]\frac{\partial}{\partial p}[u] - [\omega^*v^*]\frac{\partial}{\partial p}[v] \tag{2.19}$$
$$-\left\{\left[\frac{u^*}{a\cos\varphi}\frac{\partial}{\partial\lambda}(\phi^*)\right] + \left[\frac{v^*}{a}\frac{\partial}{\partial\varphi}(\phi^*)\right]\right\}$$
$$+\{[u^*u^*][v] - [u^*v^*][u]\}\frac{\tan\varphi}{a}$$
$$+\left[u^*g\frac{\partial}{\partial p}\tau_\lambda{}^*\right] + \left[v^*g\frac{\partial}{\partial p}\tau_\varphi{}^*\right].$$

The terms of (2.19) that arise from the pressure gradient force can be further manipulated as follows. Using the eddy form of the continuity equation and the hydrostatic equation, we can show that

$$\left[\frac{u^*}{a\cos\varphi}\frac{\partial}{\partial\lambda}(\phi^*)\right]+\left[\frac{v^*}{a}\frac{\partial}{\partial\varphi}(\phi^*)\right]$$

$$=\frac{1}{a\cos\varphi}\frac{\partial}{\partial\lambda}[u^*\phi^*]+\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}([v^*\phi^*]\cos\varphi)+\left[\phi^*\frac{\partial}{\partial p}(\omega^*)\right]$$

$$=\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}([v^*\phi^*]\cos\varphi)+\frac{\partial}{\partial p}[\omega^*\phi^*]-\left[\omega^*\frac{\partial}{\partial p}(\phi^*)\right]$$

$$=\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}([v^*\phi^*]\cos\varphi)+\frac{\partial}{\partial p}[\omega^*\phi^*]+[\omega^*\alpha^*]. \tag{2.20}$$

Substituting into the eddy kinetic energy equation, we obtain

$$\begin{aligned}
&\frac{\partial}{\partial t}KE+\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}\left\{\left([v]KE+\frac{1}{2}[v^*u^*u^*]+\frac{1}{2}[v^*v^*v^*]\right)\cos\varphi\right\}\\
&+\frac{\partial}{\partial p}\left([\omega]KE+\frac{1}{2}[\omega^*u^*u^*]+\frac{1}{2}[\omega^*v^*v^*]\right)\\
&=\frac{-[v^*u^*]}{a}\frac{\partial}{\partial\varphi}[u]-\frac{[v^*v^*]}{a}\frac{\partial}{\partial\varphi}[v]-[\omega^*u^*]\frac{\partial}{\partial p}[u]-[\omega^*v^*]\frac{\partial}{\partial p}[v]\\
&-\left\{\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}([v^*\phi^*]\cos\varphi)+\frac{\partial}{\partial p}[\omega^*\phi^*]\right\}-[\omega^*\alpha^*]\\
&+([u^*u^*][v]-[u^*v^*][u])\frac{\tan\varphi}{a}\\
&+\left[u^*g\frac{\partial}{\partial p}\tau_\lambda{}^*\right]+\left[v^*g\frac{\partial}{\partial p}\tau_\varphi{}^*\right].
\end{aligned} \tag{2.21}$$

The terms on the first line of the right-hand side represent gradient production, i.e. the conversion between the kinetic energy of the mean flow and that of the eddies. This conversion is in the sense of increasing the eddy kinetic energy when the eddy momentum flux is "down the gradient," i.e. when it is from higher mean momentum to lower mean momentum. The $\omega^*\alpha^*$ term represents eddy kinetic energy generation from eddy available potential energy, while the terms involving ϕ^* represent the effects of "pressure work."

The appearance of the metric terms in (2.21) may be somewhat surprising. They arise because we have defined "eddy" in terms of departures from the zonal mean, so that a particular latitude-longitude coordinate system is implicit in the very definition of *KE*. Obviously there cannot be any metric terms in the equation for the *total* kinetic energy, *K*.

Define the zonal kinetic energy by

$$KZ \equiv \frac{1}{2}([u]^2 + [v]^2), \tag{2.22}$$

and note that

$$[K] = KZ + KE. \tag{2.23}$$

Note that all three quantities in this equation are independent of longitude. By converting (2.6) to advective form, we find that

$$\begin{aligned}&\frac{\partial}{\partial t}[u] + \frac{[v]}{a}\frac{\partial}{\partial \varphi}\{[u]\} + [\omega]\frac{\partial}{\partial p}([u]) \\ &= -\frac{1}{a\cos\varphi}\frac{\partial}{\partial \varphi}\{[v^* u^*]\cos\varphi\} - \frac{\partial}{\partial p}[\omega^* u^*] \\ &+ ([v][u] + [v^* u^*])\frac{\tan\varphi}{a} + f[v] + g\frac{\partial}{\partial p}[\tau_\lambda].\end{aligned} \tag{2.24}$$

Multiply this result by $[u]$, to obtain

$$\begin{aligned}&\frac{\partial}{\partial t}\frac{[u]^2}{2} + \frac{[v]}{a}\frac{\partial}{\partial \varphi}\left\{\frac{[u]^2}{2}\right\} + [\omega]\frac{\partial}{\partial p}\left(\frac{[u]^2}{2}\right) \\ &= -\frac{[u]}{a\cos\varphi}\frac{\partial}{\partial \varphi}\{[v^* u^*]\cos\varphi\} - [u]\frac{\partial}{\partial p}([\omega^* u^*]) \\ &+ [u]([v][u] + [v^* u^*])\frac{\tan\varphi}{a} + f[u][v] + [u]g\frac{\partial}{\partial p}[\tau_\lambda].\end{aligned} \tag{2.25}$$

Now go back to flux form:

$$\begin{aligned}&\frac{\partial}{\partial t}\frac{[u]^2}{2} + \frac{1}{a\cos\varphi}\frac{\partial}{\partial \varphi}\left\{[v]\frac{[u]^2}{2}\cos\varphi\right\} + \frac{\partial}{\partial p}\left\{[\omega]\frac{[u]^2}{2}\right\} \\ &= -\frac{[u]}{a\cos\varphi}\frac{\partial}{\partial \varphi}\{[v^* u^*]\cos\varphi\} - [u]\frac{\partial}{\partial p}([\omega^* u^*]) \\ &+ [u]([v][u] + [v^* u^*])\frac{\tan\varphi}{a} + f[u][v] + [u]g\frac{\partial}{\partial p}[\tau_\lambda].\end{aligned} \tag{2.26}$$

Similarly, we can show that

$$\frac{\partial}{\partial t}\frac{[v]^2}{2} + \frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}\left\{[v]\frac{[v]^2}{2}\cos\varphi\right\} + \frac{\partial}{\partial p}\left\{[\omega]\frac{[v]^2}{2}\right\}$$
$$= -\frac{[v]}{a}\frac{\partial}{\partial\varphi}[\phi] - \frac{[v]}{a\cos\varphi}\frac{\partial}{\partial\varphi}\{[v^*v^*]\cos\varphi\} - [v]\frac{\partial}{\partial p}([\omega^*v^*]) \tag{2.27}$$
$$-[v]([u][u] + [u^*u^*])\frac{\tan\varphi}{a} - f[u][v] + [v]g\frac{\partial}{\partial p}[\tau_\lambda].$$

Adding (2.26) and (2.27), we obtain an equation governing the zonal kinetic energy:

$$\frac{\partial}{\partial t}KZ + \frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}\{[v]KZ\cos\varphi\} + \frac{\partial}{\partial p}\{[\omega]KZ\}$$
$$= -\frac{[v]}{a}\frac{\partial}{\partial\varphi}[\phi] - \frac{[u]}{a\cos\varphi}\frac{\partial}{\partial\varphi}\{[v^*u^*]\cos\varphi\} - [u]\frac{\partial}{\partial p}([\omega^*u^*])$$
$$-\frac{[v]}{a\cos\varphi}\frac{\partial}{\partial\varphi}\{[v^*v^*]\cos\varphi\} - [v]\frac{\partial}{\partial p}([\omega^*v^*]) \tag{2.28}$$
$$+([u][v^*u^*] - [v][u^*u^*])\frac{\tan\varphi}{a}$$
$$+[u]g\frac{\partial}{\partial p}[\tau_\lambda] + [v]g\frac{\partial}{\partial p}[\tau_\lambda].$$

We can manipulate the geopotential term of this equation by use of the zonally averaged continuity equation:

$$\frac{[v]}{a}\frac{\partial}{\partial\varphi}[\phi] = \frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}([v][\phi]\cos\varphi) + [\phi]\frac{\partial}{\partial p}[\omega]$$
$$= \frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}([v][\phi]\cos\varphi) + \frac{\partial}{\partial p}([\phi][\omega]) - [\omega]\frac{\partial}{\partial p}[\phi] \tag{2.29}$$
$$= \frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}([v][\phi]\cos\varphi) + \frac{\partial}{\partial p}([\phi][\omega]) + [\omega][\alpha].$$

This can be substituted into (2.28), to obtain *the zonal kinetic energy equation:*

$$
\begin{aligned}
&\frac{\partial}{\partial t}KZ + \frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}([v]KZ\cos\varphi) + \frac{\partial}{\partial p}([\omega]KZ) \\
&= -\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}([v][\phi]\cos\varphi) - \frac{\partial}{\partial p}([\phi][\omega]) - [\omega][\alpha] \\
&\quad -\frac{[u]}{a\cos\varphi}\frac{\partial}{\partial\varphi}\{[v^* u^*]\cos\varphi\} - [u]\frac{\partial}{\partial p}([\omega^* u^*]) \\
&\quad -\frac{[v]}{a\cos\varphi}\frac{\partial}{\partial\varphi}\{[v^* v^*]\cos\varphi\} - [v]\frac{\partial}{\partial p}([\omega^* v^*]) \\
&\quad + ([u][v^* u^*] - [v][u^* u^*])\frac{\tan\varphi}{a} \\
&\quad + [u]g\frac{\partial}{\partial p}[\tau_\lambda] + [v]g\frac{\partial}{\partial p}[\tau_\lambda].
\end{aligned}
\tag{2.30}
$$

The terms on the third and fourth lines of the right-hand side of (2.30) can be interpreted as representing the work done by the mean flow against the "forces" exerted on the mean flow by the eddies, through eddy momentum transport.

Comparison with the eddy kinetic energy equation shows that, as expected, the metric terms do not affect the zonally averaged total kinetic energy. Adding the equations for *KZ* and *KE* gives *the equation for the zonally averaged total kinetic energy*, [*K*]:

$$
\begin{aligned}
&\frac{\partial}{\partial t}[K]+\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}\left\{\left([v][K]+\frac{1}{2}[v^*u^*u^*]+\frac{1}{2}[v^*v^*v^*]\right)\cos\varphi\right\}\\
&\quad+\frac{\partial}{\partial p}\left([\omega][K]+\frac{1}{2}[\omega^*u^*u^*]+\frac{1}{2}[\omega^*v^*v^*]\right)\\
&=\frac{-[v^*u^*]}{a}\frac{\partial}{\partial\varphi}[u]-\frac{[v^*v^*]}{a}\frac{\partial}{\partial\varphi}[v]-[\omega^*u^*]\frac{\partial}{\partial p}[u]-[\omega^*v^*]\frac{\partial}{\partial p}[v]\\
&\quad-\frac{[u]}{a\cos\varphi}\frac{\partial}{\partial\varphi}\{[v^*u^*]\cos\varphi\}-[u]\frac{\partial}{\partial p}([\omega^*u^*])\\
&\quad-\frac{[v]}{a\cos\varphi}\frac{\partial}{\partial\varphi}\{[v^*v^*]\cos\varphi\}-[v]\frac{\partial}{\partial p}([\omega^*v^*])\\
&\quad-\left\{\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}([v][\phi]\cos\varphi)+\frac{\partial}{\partial p}([\phi][\omega])\right\}-[\omega][\alpha]\\
&\quad-\left\{\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}([v^*\phi^*]\cos\varphi)+\frac{\partial}{\partial p}[\omega^*\phi^*]\right\}-[\omega^*\alpha^*]\\
&\quad+\left[u^*g\frac{\partial}{\partial p}\tau_\lambda{}^*\right]+\left[v^*g\frac{\partial}{\partial p}\tau_\varphi{}^*\right]+[u]g\frac{\partial}{\partial p}[\tau_\lambda]+[v]g\frac{\partial}{\partial p}[\tau_\lambda].
\end{aligned}
\tag{2.31}
$$

The first three lines on the right-hand side of (2.31) come from the “gradient production” terms of the eddy kinetic energy equation and the terms of the zonal kinetic energy equation that represent the work done by the mean flow against the forces exerted on the mean flow by the eddies. Inspection shows that these terms can be combined, as shown below:

$$
\begin{aligned}
&\frac{\partial}{\partial t}[K] + \frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}\left\{\left([v][K] + \frac{1}{2}[v^*u^*u^*] + \frac{1}{2}[v^*v^*v^*]\right)\cos\varphi\right\} \\
&+ \frac{\partial}{\partial p}\left([\omega][K] + \frac{1}{2}[\omega^*u^*u^*] + \frac{1}{2}[\omega^*v^*v^*]\right) \\
&= -\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}\{([u][v^*u^*] + [v][v^*v^*])\cos\varphi\} \\
&-\frac{\partial}{\partial p}([u][\omega^*u^*] + [v][\omega^*v^*]) \\
&-\left\{\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}([v][\phi]\cos\varphi) + \frac{\partial}{\partial p}([\phi][\omega])\right\} - [\omega][\alpha] \\
&-\left\{\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}([v^*\phi^*]\cos\varphi) + \frac{\partial}{\partial p}[\omega^*\phi^*]\right\} - [\omega^*\alpha^*] \\
&+\left[u^*g\frac{\partial}{\partial p}\tau_\lambda{}^*\right] + \left[v^*g\frac{\partial}{\partial p}\tau_\varphi{}^*\right] + [u]g\frac{\partial}{\partial p}[\tau_\lambda] + [v]g\frac{\partial}{\partial p}[\tau_\lambda].
\end{aligned}
\tag{2.32}
$$

Here the terms mentioned above have been combined on the first two lines of the right-hand side, and it is apparent that they take the form of a divergence. This implies, of course, that the two terms together integrate to zero over the whole atmosphere. The interpretation of this result is that the gradient production terms represent conversion between the kinetic energy of the mean flow and the kinetic energy of the eddies. We do not see simple cancellation between the corresponding terms of the *KE* and *KZ* equations, because the conversion process is not local; it occurs over a region that is extended in the meridional and vertical directions. It is only when we integrate over the whole atmosphere that the cancellation becomes apparent.