Dimensional Analysis, Scale Analysis, and Similarity Analysis

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13.1 Introduction

In part because no adequate theory of turbulence exists, various kinds of dimensional arguments are very common in the literature on the planetary boundary layer. They also have applications in the theories of planetary circulations, and in engineering. Unfortunately, dimensional analysis and related techniques are too often used in the classroom without an adequate explanation of why they work or what they are good for.

The basic premise of dimensional analysis is that physical principles must be independent of the choice of units. For example, Newton's law $F = ma$, i.e. force equals mass times acceleration, must predict the same physical phenomena whether we use English or SI units.

To discuss the concepts of dimensional analysis, we need some definitions:

1) *Standard*: an arbitrary reference measure adopted for purposes of communication. Example: meter.

2) *Physical quantity*: a conceptual property of a physical system, which can be expressed numerically in terms of one or more standards. Example: Radius of the Earth.

3) *Unit*: an arbitrary fraction or multiple of a standard, used to avoid large (or small) numbers. Example: Kilometer.

4) *Primary quantities:* a set of quantities (hereafter called q) chosen arbitrarily for the description of a problem, subject to the constraint that the units of measurement chosen for the quantities can be assigned independently. Primary quantities are sometimes called "fundamental quantities." Example: Length, time, mass.

5) *Dimension*: the relationship of a derived quantity to whatever primary quantities have been selected. Example: Velocity = Length / Time.

6) *Extraneous standard*: a standard which is irrelevant for a particular

problem. Example: Length of Henry VIII's foot.

7) *Extraneous unit*: a unit based on an extraneous standard. Example: Mile = 5280 ft.

8) *Dimensionless quantity*: a quantity which is not expressed in *extraneous* units; i.e. on which is expressed in units derivable from the problem. Notice that there are no *intrinsically* dimensionless quantities. A quantity is dimensionless or not only with respect to a particular problem. Example: Rossby number.

9) *Dimensional analysis:* the process of removing extraneous information from a problem by forming dimensionless groups.

10) *Similar systems*: those for which the dimensionless quantities have identical values, even though the dimensional quantities may differ widely in value.

13.2 The Buckingham Pi Theorem

The fundamental theorem of dimensional analysis is due to Buckingham, and is stated here without proof:

If the equation

$$\phi(q_1, q_2, \ldots, q_n) = 0 \tag{13.1}$$

is the only relationship among the q_i, and if it holds for any arbitrary choice of the units in which $q_1, q_2, \ldots, q_n$ are measured, then (13.1) can be written in the form

$$\phi'(\pi_1, \pi_2, \ldots, \pi_m) = 0 , \tag{13.2}$$

where $\pi_1, \pi_2, \ldots, \pi_m$ are independent dimensionless products of the q's.

Further, if k is the minimum number of primary quantities necessary to express the dimensions of the q's, then

$$m = n - k . \tag{13.3}$$

Since $k > 0$, $m < n$. According to (13.3), the number of dimensionless products is the number of dimensional variables minus the number of primary quantities.

Another way of writing (2) is

$$\phi'(\pi_1, \pi_2, \ldots, \pi_m; 1, \ldots, 1) = 0 \tag{13.4}$$

where the number of "1's" appearing in the argument list is k. Clearly the 1's carry no information about the functional relationship among the π's, so that we can just omit them, as was done in (2). In (4), the 1's clearly represent "extraneous" information, which entered the problem through extraneous units of the q's.

The choice of the q's can be made by inspection of the governing equations (if known), or by intuition.

The dimensions of the q's can be determined in terms of chosen primary quantities. Although the primary quantities can be chosen arbitrarily, provided that their units can be assigned independently, we must be sure to choose enough of them so that we can complete the non-dimensionalization. In atmospheric science, mass (M), length (L), time (t), and temperature (T) will almost always be enough. Very often, T and/or M can be omitted from the list. There has been a debate about whether temperature should be included as a primary quantity independent of mass, length, and time (Huntley, 1967). The answer seems to be application-dependent. In atmospheric science, temperature can be considered a fourth primary quantity. In some other fields it is useful to add electrical charge to the list of primary quantities. There are further possibilities and issues, but they will not be discussed here because they are not relevant to atmospheric science.

13.3 Example of dimensional analysis

Consider thermal convection in a laboratory tank. The six variables of the problem are

$$g, \Gamma, h, \nu, \kappa, \alpha \, .$$

Here g is gravity, Γ is the lapse rate, h is the depth of the fluid, ν is the molecular viscosity, κ is the molecular thermal conductivity, and α is a parameter which measures the amount of thermal expansion per unit temperature change. As primary quantities, we choose length (L), time (t), and temperature (T). There are three primary quantities, and six variables, so we should be able to eliminate 6 - 3 = 3 pieces of extraneous information. We can tabulate the dimensions of the q's as follows:

Quantity	Dimensions		
	L	t	T
g	1	-2	0
α	0	0	-1
Γ	-1	0	1

h	1	0	0
ν	2	-1	0
κ	2	-1	0

As our pertinent (non-extraneous) unit of length, we choose h. Forming products, we systematically eliminate the "lengths" from our set of quantities:

Quantity	Dimensions		
	L	t	T
gh^{-1}	0	-2	0
α	0	0	-1
Γh	0	0	1
1	0	0	0
νh^{-2}	0	-1	0
κh^{-2}	0	-1	0

As our unit of time, we take $h^2\nu^{-1}$. (Notice that we could just as well take $h^2\kappa^{-1}$.) Again forming products, we obtain

Quantity	Dimensions		
	L	t	T
$gh^3\nu^{-2}$	0	0	0
α	0	0	-1
Γh	0	0	1
1	0	0	0
1	0	0	0
$\kappa\nu^{-1}$	0	0	0

Obviously, to complete the procedure, we simply form the product $\Gamma\alpha h$. All together, we then have three 1's - three pieces of extraneous information have been eliminated, as promised by the π-theorem. But we have the *three* nondimensional combinations

$$gh^3\nu^{-2} \equiv x_1 , \tag{13.5}$$

$$\kappa\nu^{-1} \equiv Pr , \tag{13.6}$$

and

$$\Gamma\alpha h \equiv x_2 . \tag{13.7}$$

Notice that

$$Ra = Prx_1x_2 \tag{13.8}$$

so that we can regard Pr, Ra, and x_1 (say) as our three combinations.

In the study of Rayleigh convection without shear, (e.g. Chandraskhar, 1961) it is demonstrated that only *two* dimensionless combinations matter: Pr and Ra. Why have we found three? In the governing equations, g and α appear only in the combination $g\alpha$, so that they need not be separately included in our list of q's. Because this reduces the number of q's by one, without reducing the number primary quantities, the π-theorem tells us that the number of dimensionless combinations will also be reduced by one.

13.4 Similarity

"Similar systems" were defined earlier as those for which the dimensionless combinations have identical values, even though the dimensional quantities may be quite different. For example, two laboratory Rayleigh convection experiments, may be performed in which ν, κ, and Γ are all different. But if Pr and Ra are the same, then the (nondimensionalized) results will be the same; convection will set in at the same critical Ra, even though the critical Γ's will be different. If the data from many "similar" experiments are plotted in nondimensional form, all of the data will fall neatly onto families of curves. For example, for a given value of Pr, we can plot the nondimensional heat flux in Rayleigh convection experiments against Ra. For each value of Pr, the data will fall onto orderly curves. If the same data were plotted in dimensional form, a "scattered" set of points would result; no order would be apparent. In short, dimensional analysis is useful for the organization of data.

A "similarity hypothesis" or "similarity theory," asserts that one or more dimensionless groups can be omitted (neglected) from the list of groups needed for a complete description of a problem. The hypothesis can be tested by plotting experimental results in non-dimensional form, without regard for the omitted groups. If no appreciable scatter of the data occurs, then the hypothesis is confirmed.

A similarity analysis does not *explain* anything, however, in the sense that it does not allow us to deduce the behavior of a specific physical system from a more general set of physical laws; it only helps us to organize our data. For this reason, the term "similarity theory" is a misnomer, if we insist that, by definition, theories *explain*.

Similarity analyses can proceed without knowledge of the governing equations. In fact, when the governing equations are known, we proceed by "scale analysis," in which nondimensionalization is used to compare the magnitudes of various terms of the equations. For this reason, it seems best to reserve the term "similarity analysis" for those studies in which the governing equations are not known.

13.5 *Example of similarity analysis*

A famous example of similarity analysis was provided by G.I. Taylor (1950 a, b), who analyzed nuclear blasts. He chose the following q's for the problem:

Table 4.1:

Symbol	*Definition*	*Representative value or first guess*
R	radius of wavefront	10^2 m
t	time	10^{-2} s
p_0	ambient pressure	10^5 Pa
ρ_0	ambient density	1 kg m^{-3}
E	energy released	10^{14} J

From these, he arrived at the followings π's:

$$\frac{\rho_0 R^5}{Et^2}, \frac{p_0 R^3}{E}. \tag{13.9}$$

With the numerical values given in the table above, including a "first guess" at the value of E, we find that

$$\frac{\rho_0 R^5}{Et^2} = 1, \tag{13.10}$$

$$\frac{p_0 R^3}{E} = 10^{-3}. \tag{13.11}$$

Because the second parameter is much less than one, Taylor concluded that it was physically irrelevant. The second parameter involves the ambient pressure, so this amounts to the assumption that the nuclear fireball does not give a damn about the ambient pressure. This similarity hypothesis led Taylor to conclude that

$$\frac{\rho_0 R^5}{Et^2} = A = \text{constant} = 1. \tag{13.12}$$

[compare with (13.10)]. He could then use (13.12) to evaluate E from knowledge of $R(t)$, based on published (unclassified) magazine photos of the explosion. His accurate estimate of E was embarrassing to the government, which had not declassified that information.

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