Vertical Coordinate Transformations

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Consider two vertical coordinates, denoted by z and z', respectively. Although the "z" symbol suggests height, no such implication is intended here; z and z' can be any variables at all so long as they vary monotonically with height. Suppose that we have in hand a rule telling how to compute z' for a given value of z, and vice versa. For example, we might define $z' \equiv z - z_S(x, y, t)$, where $z_S(x, y, t)$ is the value of z at the Earth's surface.

Let x denote an independent variable, which can be either a horizontal coordinate or time. Consider the variation of an arbitrary dependent variable, A, with x, as sketched in Fig. 1. Note that $\delta x \equiv x_2 - x_1$ and $\delta z' \equiv z'_3 - z'_2 = z'_1 - z'_2$. Our goal here is to relate $\left(\frac{\partial A}{\partial x}\right)_{z'}$ to $\left(\frac{\partial A}{\partial x}\right)_z$. With reference to Fig. 1, we can write

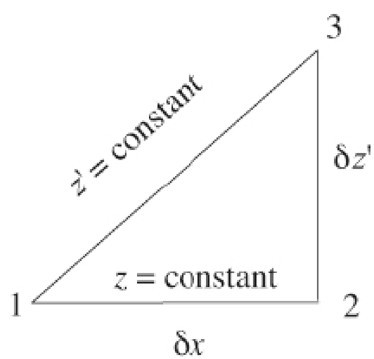

Figure 1: Sketch used to derive the chain rule relating derivatives on surfaces of constant z to those on surfaces of constant z'.

$$\frac{A_3 - A_1}{x} = \frac{A_2 - A_1}{x} + \left(\frac{A_3 - A_2}{x}\right)$$

$$= \frac{A_2 - A_1}{x} + \left(\frac{A_3 - A_2}{x}\right)\frac{z'}{x} \tag{1}$$

$$= \frac{A_2 - A_1}{x} + \left(\frac{A_3 - A_2}{x} \right) \left(\frac{z'_1 - z'_2}{x_2 - x_1} \right)$$

$$= \frac{A_2 - A_1}{x} - \left(\frac{A_3 - A_2}{z'_3 - z'_2} \right) \left(\frac{z'_2 - z'_1}{x_2 - x_1} \right) .$$

Taking the limit as the increments become small, we obtain

$$\left(\frac{\partial A}{\partial x} \right)_{z'} = \left(\frac{\partial A}{\partial x} \right)_{z} - \left(\frac{\partial A}{\partial z'} \right)_{x} \left(\frac{\partial z'}{\partial x} \right)_{z} . \tag{2}$$

In the example suggested above, with $z' \equiv z - z_S(x, y, t)$, Eq. 2 would reduce to

$$\left(\frac{\partial A}{\partial x} \right)_{z'} = \left(\frac{\partial A}{\partial x} \right)_{z} - \left(\frac{\partial A}{\partial z'} \right)_{x} \frac{\partial z_S}{\partial x} . \tag{3}$$