Virtual Temperature and Virtual Temperature Fluxes

David Randall

Definition of virtual temperature

For dry air, the equation of state is

$$p_d = \rho_d R_d T.$$  \hspace{1cm} (1)

Similarly, water vapor obeys its own equation of state with its own gas constant:

$$e = \rho_v R_v T.$$  \hspace{1cm} (2)

The total pressure is

$$p = p_d + e.$$  \hspace{1cm} (3)

We see that

$$p = (\rho_d R_d + \rho_v R_v)T,$$  \hspace{1cm} (4)

$$\rho = \rho_d + \rho_v + \rho_l.$$  \hspace{1cm} (5)

We define the virtual temperature as satisfying the ideal gas law with the total pressure and the total density, and the gas constant for dry air:

$$\rho R_d T_v \equiv p.$$  \hspace{1cm} (6)

Substituting on both sides of (6), we obtain
\[(\rho_d + \rho_v + \rho_l)R_dT_v = (\rho_dR_d + \rho_vR_v)T \]
\[= \left(\rho_d + \frac{\rho_v}{\varepsilon}\right)R_dT, \]  
(7)

where
\[\varepsilon = \frac{R_d}{R_v} \approx 0.622. \]  
(8)

This leads to

\[T_v = T \left(\frac{1 + \frac{q}{\varepsilon}}{1 + \frac{q}{\varepsilon} + l}\right) \]
(9)

where \(q \equiv \frac{\rho_v}{\rho_d}\) and \(l \equiv \frac{\rho_l}{\rho_d}\) are the mixing ratios of water vapor and dry air, respectively. Eq. (9) can be approximated by

\[T_v \equiv T(1 + \delta q - l), \]
(10)

where
\[\delta \equiv \frac{1 - \varepsilon}{\varepsilon} \approx 0.608. \]  
(11)

Buoyancy fluctuations on isobaric surfaces

The following discussion is based on ideas presented by Lilly (1968). The virtual dry static energy is

\[s_v \equiv c_pT_v + gz. \]
(12)

We use the moist static energy

\[h \equiv c_pT + gz + Lq, \]
(13)
which is approximately conserved under both moist and dry adiabatic processes, even when precipitation is occurring. The total water mixing ratio, \( q + l' \), is also approximately conserved under both moist and dry adiabatic processes, although it is of course affected by precipitation.

Consider fluctuations at constant pressure, denoted by primes. We can write

\[ h' \equiv c_p T' + Lq'. \]  

(14)

Here we neglect height fluctuations on the isobaric surfaces. From (10) and (12), we see that

\[ s'_v \equiv c_p T'_v \]

\[ = c_p T' + c_p \bar{T} (\delta q' - l') \]

\[ = c_p T' + \epsilon (\delta Lq' - Ll') , \]  

(15)

where

\[ \epsilon \equiv \frac{c_p \bar{T}}{L} \equiv 0.1 . \]  

(16)

This can be manipulated as follows:

\[ s'_v \equiv (c_p T' + Lq') - (1 - \delta \epsilon) Lq' - \epsilon Ll' \]

\[ = h' - (1 - \delta \epsilon) L(q' + l') + \left[ 1 - (1 + \delta) \epsilon \right] Ll' . \]  

(17)

The coefficients \( 1 - \delta \epsilon \) and \( 1 - (1 + \delta) \epsilon \) are both positive and nondimensional.

**Buoyancy fluxes in clear and uniformly cloudy layers**

Following Lilly (1968), an expression for the virtual dry static energy flux, \( F_{sv} \), can be obtained by multiplying (17) by \( \rho w' \) and then averaging. (We neglect the fluctuations of \( \rho \).) The result is

\[ F_{sv} = F_h - (1 - \delta \epsilon) LF_{q+1} + \left[ 1 - (1 + \delta) \epsilon \right] LF_j . \]  

(18)

Eq. (18) is valid regardless of the cloud amount. It was used by Randall (1987). According to (18), for given values of \( F_h \) and \( F_{q+1} \), the buoyancy flux increases as the liquid water flux increases.
Still following Lilly (1968), we consider two cases. First, if there is no cloud, then \( LF_i = 0 \), and (18) reduces to
\[
F_{sw} = (F_{sw})_{cl} \equiv F_h - (1 - \delta c) LF_{q+l} \quad \text{for clear air.}
\]
(19)

If there is a uniform cloud, so that the air is saturated everywhere, we can write
\[
\gamma c_p' T' \equiv Lq' \quad \text{in a uniform cloud,}
\]
(20)

where
\[
\gamma \equiv \frac{L}{c_p} \left( \frac{\partial q_c}{\partial T} \right)_p.
\]
(21)
From (20), we get
\[
\gamma h' = (1 + \gamma) Lq' \quad \text{in a uniform cloud,}
\]
(22)
or
\[
Lq' = \left( \frac{\gamma}{1 + \gamma} \right) h' \quad \text{in a uniform cloud.}
\]
(23)
It follows that
\[
Ll' = L(q' + l') - \left( \frac{\gamma}{1 + \gamma} \right) h' \quad \text{in a uniform cloud,}
\]
(24)
or
\[
LF_i = (LF_i)_{cl} \equiv LF_{q+l} - \left( \frac{\gamma}{1 + \gamma} \right) F_h \quad \text{in a uniform cloud.}
\]
(25)
Substituting (25) into (18), we find that for the fully cloudy case
\[
F_{sv} = (F_{sv})_{cl} \equiv \beta F_h - \varepsilon LF_{q+l}
\]
\text{in a uniform cloud,}
\tag{26}

where
\[
\beta \equiv \frac{1 + (1 + \delta)\gamma \varepsilon}{1 + \gamma}.
\tag{27}
\]

The notations \((LF_i)_{cl}\) and \((F_{sv})_{cl}\), defined in (25) and (26) respectively, will be used below.

**Buoyancy reversal**

The fluxes just below the PBL top satisfy
\[
(F_h)_B = -E \Delta h + \Delta R, \tag{28}
\]
\[
(F_{q+l})_B = -E \Delta (q + l), \tag{29}
\]
and
\[
(F_{sv})_B = -E \left[ \beta \Delta h - \varepsilon L \Delta (q + l) \right] + \beta \Delta R \text{ in a uniform cloud.} \tag{30}
\]

We want to rewrite (30) in the form
\[
(F_{sv})_B = -E \left[ \Delta s_v - (\Delta s_v)_{\text{crit}} \right] + \beta \Delta R \text{ for a uniform cloud.} \tag{31}
\]

We can also use (31) a cloud-free boundary layer, if we set \((\Delta s_v)_{\text{crit}} = 0\) and \(\Delta R = 0\). To find an expression for \((\Delta s_v)_{\text{crit}}\) that can be used for boundary layers that are uniformly cloud just below the inversion, we use (15) to write
\[
\Delta s_v \equiv \Delta s + \delta \varepsilon L \Delta q - \varepsilon L \Delta l
\] 
\[
= \Delta s + \delta \varepsilon L \Delta q - \varepsilon L \Delta (q + l) + \varepsilon L \Delta q
\]
\[
= \Delta s + (1 + \delta) \varepsilon L \Delta q - \varepsilon L \Delta (q + l)
\]
\[
= \Delta h - \left[ 1 - (1 + \delta) \varepsilon \right] L \Delta q - \varepsilon L \Delta (q + l).
\tag{32}
\]
Eq. (32) is valid whether or not a cloud is present. For the special case of a uniform cloud, the water vapor jump can be written as

\[ L \Delta q = L (q_{B^+} - q_B) \]
\[ = L (q_{B^+} - q_{B^*}) \]
\[ = L (q_{B^*} - q_{B^*}) - L (q_{B^*} - q_{B^+}) \]
\[ = \gamma \Delta s - L (q_{B^*} - q_{B^+}) \text{ for the case of a uniform cloud.} \]

(33)

From (33), we see that

\[ (1 + \gamma) L \Delta q = \gamma \Delta h - L (q_{B^*} - q_{B^+}) \text{ for a uniform cloud,} \]

(34)

or

\[ L \Delta q = \frac{\gamma \Delta h - L (q_{B^*} - q_{B^+})}{(1 + \gamma)} \text{ for a uniform cloud.} \]

(35)

Substituting (35) back into (32), we find that

\[ \Delta s_v \equiv \Delta h - \left[ 1 - (1 + \delta) \epsilon \right] \frac{\gamma \Delta h - L (q_{B^*} - q_{B^+})}{(1 + \gamma)} - \epsilon L \Delta (q + l) \]
\[ = \left[ 1 - \frac{1 - (1 + \delta) \epsilon}{1 + \gamma} \right] \Delta h - \epsilon L \Delta (q + l) + \left[ \frac{1 - (1 + \delta) \epsilon}{1 + \gamma} \right] L (q_{B^*} - q_{B^+}) \]
\[ = \beta \Delta h - \epsilon L \Delta (q + l) + \left[ \frac{1 - (1 + \delta) \epsilon}{1 + \gamma} \right] L (q_{B^*} - q_{B^+}) \text{ for a uniform cloud.} \]

(36)

Eq. (36) can be rearranged to

\[ \beta \Delta h - \epsilon L \Delta (q + l) = \Delta s_v - \left[ \frac{1 - (1 + \delta) \epsilon}{1 + \gamma} \right] L (q_{B^*} - q_{B^+}) \text{ for a uniform cloud.} \]

(37)

Comparing (30), (31), and (36), we see that
\[
(\Delta s)_\text{crit} = \left[ \frac{1-(1+\delta)e}{1+\gamma} \right] L(q_{B+} - q_{B+})
\]
for a uniform cloud.

(38)

**Buoyancy fluxes in partly cloudy layers**

Eq. (18) can be used to determine the buoyancy flux in partly cloudy layers, following Randall (1987), but with some changes in notation. We use a mass flux approach for all fluxes, e.g.,

\[
F_\psi = M_c (\psi_u - \psi_d),
\]

(39)

where \(\psi\) is a generic intensive variable,

\[
M_c \equiv \rho \sigma (1-\sigma)(w_u-w_d),
\]

(40)

\[
\psi' = \sigma \psi_u + (1-\sigma) \psi_d,
\]

(41)

and \(\sigma\) is the fractional area covered by rising motion. From (41), we see that

\[
\psi_u = \psi' + (1-\sigma)(\psi_u - \psi_d) \quad \text{and} \quad \psi_d = \psi' - \sigma(\psi_u - \psi_d).
\]

(42)

Eq. (42) plays an important role in the analysis below.

In order to use (18), we have to determine the flux of liquid water. The liquid water mixing ratio in the updraft is given by

\[
l_u = \text{Max}\left\{ (q+l)_u - (q_s)_u, 0 \right\} \\
= \text{Max}\left\{ \left[ (q+l)_u - (q+\bar{l}) \right] + \left[ (q+\bar{l}) - \bar{q}_s \right] + \left[ \bar{q}_s - (q_s)_u \right], 0 \right\},
\]

(43)

where \(\bar{q}_s\) is the saturation mixing ratio at temperature \(\bar{T}\). The second line of (43) shows that, for a given mean state, the liquid water in the updraft depends on three things:

- the total water difference between the updraft and the mean state, \((q+l)_u - (q+\bar{l})\),
- the relative humidity of the mean state itself, as measured by \((q+\bar{l}) - \bar{q}_s\), and
the difference in saturation mixing ratio between the mean state and the updraft, as 
measure by $\overline{q} - (q_u)$.

The difference in saturation mixing ratios can be written as

$$L[\overline{q} - (q_u)] = \left( \frac{\gamma}{1 + \gamma} \right) [\overline{h} - (h_u)]$$

(44)

Here we have replaced $(h_u)$ by $h_u$, which is only correct when the updraft is saturated. 

Substituting (44) into (43), we obtain

$$LL_u = \text{Max}\left\{ L\tilde{l} + L[(q + l)_u - (q + l)] - \left( \frac{\gamma}{1 + \gamma} \right)(h_u - \overline{h}), 0 \right\},$$

(45)

where

$$L\tilde{l} \equiv L[\overline{q} - \overline{q} - \left( \frac{\gamma}{1 + \gamma} \right)(\overline{h} - \overline{h})]$$

(46)

is a property of the mean state. Finally, we use (42) to rewrite (45) as

$$LL_u = \text{Max}\left\{ L\tilde{l} + (1 - \sigma)L[(q + l)_u - (q + l)_d] - \left( \frac{\gamma}{1 + \gamma} \right)(1 - \sigma)(h_u - h_d), 0 \right\}.$$

(47)

In a similar way, we can show that

$$LL_d = \text{Max}\left\{ L\tilde{l} - \sigma L[(q + l)_u - (q + l)_d] + \left( \frac{\gamma}{1 + \gamma} \right)\sigma(h_u - h_d), 0 \right\}.$$

(48)

To interpret the meaning of $\tilde{l}$, consider some particular cases. When both updraft and downdraft are saturated, we get
This means that when both the updraft and downdraft are saturated \( \tilde{l} \) is simply equal to the mean liquid water mixing ratio. A similar conclusion can be drawn directly from (46): When the mean state is saturated, we have \( (q+l) - \bar{q}_u = \tilde{l} \) and \( h_u - \bar{h} = 0 \), so that (46) reduces to \( \tilde{l} \equiv \tilde{l} \). If only the updraft is saturated, it is possible for \( \tilde{l} \) to be negative. In that case, \( \tilde{l} \) can be interpreted as the amount of liquid water that would have to be isobarically evaporated into the mean state in order to bring it to saturation. In general, \( \tilde{l} \) is a measure of the relative humidity of the mean state.

We now see that Eq. (47) expresses the liquid water content of the updraft in terms of a measure of the relative humidity of the mean state, namely \( \tilde{l} \), and the updraft-downdraft differences in \( q+l \) and \( h \), weighted by \( 1-\sigma \). If the updraft-downdraft differences are set to zero, then (47) reduces to \( l_u = \tilde{l} \equiv \tilde{l} \).

Similar comments apply to (48).

According to (47), \( l_u \) increases as \( \sigma \) decreases, for a given mean state and given updraft-downdraft differences in \( h \) and \( q+l \). Why should this be true? From (42), we see that for a generic variable

\[
\psi_u - \bar{\psi} = -\left(\frac{1-\sigma}{\sigma}\right)(\psi_d - \bar{\psi}).
\]

(50)

This means that as \( \sigma \) decreases towards zero, the updraft properties become more different from those of the mean, and of course the downdraft properties become closer to those of the mean. For a given mean state and given updraft-downdraft differences in \( h \) and \( q+l \), the wettest possible updraft is obtained in the limit as \( \sigma \to 0 \).

Now define
\[ L(F_i)_u = M_c L l_u \geq 0 , \]  
(51)

and

\[ L(F_i)_d \equiv -M_c L l_d \leq 0 . \]  
(52)

With these definitions, the total liquid water flux can be expressed as

\[ LF_i = M_c L (l_u - l_d) = L(F_i)_u + L(F_i)_d . \]  
(53)

Using (47), we can now write

\[
L(F_i)_u = M_c L l_u \\
= M_c \text{Max} \left\{ L\tilde{l} + (1 - \sigma) L[(q + l)_u - (q + l)_d] - (1 - \sigma) \left( \frac{\gamma}{1 + \gamma} \right) (h_u - h_d), 0 \right\} \\
= \text{Max} \left\{ M_c L\tilde{l} + (1 - \sigma) L(F_i)_{\text{clld}}, 0 \right\} .
\]  
(54)

Here we have used the \((LF_i)_{\text{clld}}\) notation defined in (25). Similarly, we find that

\[
L(F_i)_d \equiv -M_c L l_d \\
= -M_c \text{Max} \left\{ L\tilde{l} - \sigma L[(q + l)_u - (q + l)_d] + \left( \frac{\gamma}{1 + \gamma} \right) \sigma (h_u - h_d), 0 \right\} \\
= -\text{Max} \left\{ M_c L\tilde{l} - \sigma (LF_i)_{\text{clld}}, 0 \right\} .
\]  
(55)

Suppose now that the updraft contains liquid water but the downdraft does not. In that case, the total liquid water flux satisfies

\[
L(F_i) = M_c L\tilde{l} + (1 - \sigma) (LF_i)_{\text{clld}} \quad \text{for partly cloudy layers}.
\]  
(56)

Recall that \((LF_i)_{\text{clld}}\) is the liquid water flux in a uniformly cloudy layer. It is therefore surprising to see that \((LF_i)_{\text{clld}}\) is weighted, in (56), by \(1 - \sigma\), which is the fraction of the area that is not cloudy.
Substituting (56) into (18), we obtain a similarly strange result:

\[
F_{sv} = F_h - (1 - \delta e) LF_{q+1} + [1 - (1 + \delta) e] [ M_c L \tilde{L} + (1 - \sigma) (LF_{cld}) ] \\
= \sigma [ F_h - (1 - \delta e) LF_{q+1} ] \\
+ (1 - \sigma) [ F_h - (1 - \delta e) LF_{q+1} + [1 - (1 + \delta) e] (LF_{cld}) ] \\
+ [1 - (1 + \delta) e] M_c L \tilde{L}
\]

(57)

or

\[
F_{sv} = \sigma (F_{sv})_{clr} + (1 - \sigma) (F_{sv})_{cld} + [1 - (1 + \delta) e] M_c L \tilde{L}
\]

for partly cloudy layers.

(58)

According to (58), the total buoyancy flux in a partly cloudy layer involves a weighted sum of \((F_{sv})_{clr}\) and \((F_{sv})_{cld}\), as might be expected (e.g., Sommeria and Deardorff, 1977), but the weights are “backward,” in that \((F_{sv})_{clr}\) is weighted by the cloud fraction, and \((F_{sv})_{cld}\) is weighted by one minus the cloud fraction. When I first derived (58), I thought that I had made an algebra mistake. The contribution from \([1 - (1 + \delta) e] M_c L \tilde{L}\) is expected to be negative in most cases.

It can also be useful to write the buoyancy flux in partly cloudy layers as
\[ F_{sv} = \sigma \left[ F_h - (1 - \delta \epsilon) LF_{q+1} \right] \]
\[ + (1 - \sigma) \left[ F_h - (1 - \delta \epsilon) LF_{q+1} + [1 - (1 + \delta) \epsilon] (LF)_{cl} \right] \]
\[ + [1 - (1 + \delta) \epsilon] M_c L \]
\[ = \sigma \left[ F_h - (1 - \delta \epsilon) LF_{q+1} \right] \]
\[ + (1 - \sigma) \left\{ F_h - (1 - \delta \epsilon) LF_{q+1} + [1 - (1 + \delta) \epsilon] \left[ LF_{q+1} - \left( \frac{\gamma}{1 + \gamma} \right) F_h \right] \right\} \]
\[ + [1 - (1 + \delta) \epsilon] M_c L \]
\[ = F_h \left\{ \sigma + (1 - \sigma) - (1 - \sigma) \left[ 1 - (1 + \delta) \epsilon \right] \left( \frac{\gamma}{1 + \gamma} \right) \right\} \]
\[ + LF_{q+1} \left\{ -\sigma (1 - \delta \epsilon) - (1 - \sigma) (1 - \delta \epsilon) + (1 - \sigma) \left[ 1 - (1 + \delta) \epsilon \right] \right\} \]
\[ + [1 - (1 + \delta) \epsilon] M_c L \]
\[ = F_h \left\{ 1 - (1 - \sigma) \left[ 1 - (1 + \delta) \epsilon \right] \left( \frac{\gamma}{1 + \gamma} \right) \right\} + LF_{q+1} \left\{ -\left( 1 - \delta \epsilon \right) + (1 - \sigma) \left[ 1 - (1 + \delta) \epsilon \right] \right\} \]
\[ + [1 - (1 + \delta) \epsilon] M_c L \]
\[ = F_h \left\{ 1 - (1 - \sigma) \left[ 1 - (1 + \delta) \epsilon \right] \left( \frac{\gamma}{1 + \gamma} \right) \right\} + LF_{q+1} \left\{ -\left( 1 - \delta \epsilon \right) + (1 - \sigma) \left[ 1 - (1 + \delta) \epsilon \right] \right\} \]
\[ + [1 - (1 + \delta) \epsilon] M_c \left\{ L \left[ (q+l)_u - \bar{q}_u \right] - \left( \frac{\gamma}{1 + \gamma} \right) (\bar{h}_u - \bar{h}) \right\} \]

\[ (59) \]

It follows that

\[ \left[ (s_v)_u - (s_v)_d \right] = \left( h_u - h_d \right) \left\{ 1 - (1 - \sigma) \left[ 1 - (1 + \delta) \epsilon \right] \left( \frac{\gamma}{1 + \gamma} \right) \right\} \]
\[ + L \left[ (q+l)_u - (q+l)_d \right] \left\{ - (1 - \delta \epsilon) + (1 - \sigma) \left[ 1 - (1 + \delta) \epsilon \right] \right\} \]
\[ + [1 - (1 + \delta) \epsilon] \left\{ L \left[ (q+l) - \bar{q}_s \right] - \left( \frac{\gamma}{1 + \gamma} \right) (\bar{h}_s - \bar{h}) \right\} \]

\[ (60) \]
References and Bibliography


