
Inertial Stability and Instability

David Randall

Introduction

Inertial stability and instability are relevant to the atmosphere and ocean, and also in other contexts such as engineering. We will approach the problem from multiple points of view.

An engineering perspective

The first discussion is based on the paper of John Strutt (1916), also known as “Lord Rayleigh.” I have simplified the discussion, and modified the notation and terminology to be more consistent what we use in atmospheric science.

We adopt cylindrical coordinates (λ, r) , with azimuthal and radial velocity components (u, v) . To avoid unnecessary complications, the density is considered to be a constant, denoted by ρ_0 . We assume from the beginning that there are no variations in the azimuthal (λ) direction. Define the Lagrangian time derivative by

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + v \frac{\partial}{\partial r} . \tag{1}$$

The equations of motion can then be written as

$$\frac{Du}{Dt} + \frac{uv}{r} = 0 , \tag{2}$$

$$\frac{Dv}{Dt} - \frac{u^2}{r} = -\frac{\partial}{\partial r} \left(\frac{p}{\rho_0} \right) , \tag{3}$$

where p is the pressure. No pressure-gradient term appears in (2), because we have assumed that $\frac{\partial}{\partial \lambda} = 0$. Notice that (2) and (3) do not include the Coriolis terms, because we are not using a

rotating frame of reference. These equations are relevant to machinery, but are not directly applicable to the atmosphere.

Eq. (2) can be rewritten as

$$\frac{DM}{Dt} = 0, \quad (4)$$

where

$$M \equiv ru \quad (5)$$

is the angular momentum about the origin, per unit mass, and we have used

$$v \equiv \frac{Dr}{Dt}. \quad (6)$$

According to (4), the angular momentum of a particle is conserved. The reason is that there is no azimuthal pressure gradient.

For the balanced state with $Dv/Dt = 0$, Eq. (3) can be written as

$$\frac{M^2}{r^3} = \frac{\partial}{\partial r} \left(\frac{p}{\rho_0} \right). \quad (7)$$

In geophysical parlance, this is a statement of *cyclostrophic balance*. There is an analogy between cyclostrophic balance and hydrostatic balance. In hydrostatic balance, , i.e., when the vertical acceleration of the air is negligible, the pressure changes with height as required to balance the weight of the air above. Similarly, according to (7), when $Dv/Dt = 0$ (required for cyclostrophic balance), the pressure changes with radius as required to balance the centrifugal acceleration, u^2/r . As you know, hydrostatic balance can be either stable or unstable, depending on the rate of change of temperature with height. Similarly, cyclostrophic balance can be either stable or unstable, depending on the rate of change of M with r .

We now linearize our equations about a cyclostrophically balanced basic state. The basic state is denoted by an overbar and the perturbations by primes. The basic state satisfies $\bar{v} = 0$ and $\frac{\bar{u}^2}{r} = \frac{\partial}{\partial r} \left(\frac{\bar{p}}{\rho_0} \right)$. The linearized version of (3) can be written as

$$\frac{\partial v'}{\partial t} - \frac{2\bar{M}}{r^3} M' = -\frac{\partial}{\partial r} \left(\frac{p'}{\rho_0} \right), \quad (8)$$

and the linearized version of (4) is

$$\frac{\partial M'}{\partial t} + v' \frac{\partial \bar{M}}{\partial r} = 0. \quad (9)$$

For simplicity, we ignore the pressure-gradient term of (8), which plays only a secondary role. We look for solutions of the form $(v', M') = (\hat{v}, \hat{M}) e^{\sigma t}$. Then our system reduces to

$$\sigma \hat{v} - \frac{2\bar{M}}{r^3} \hat{M} = 0, \quad (10)$$

$$\frac{\partial \bar{M}}{\partial r} \hat{v} + \sigma \hat{M} = 0. \quad (11)$$

For nontrivial solutions, we need

$$\sigma^2 + \frac{2\bar{M}}{r^3} \frac{\partial \bar{M}}{\partial r} = 0, \quad (12)$$

or

$$\sigma^2 = -\frac{1}{r^3} \frac{\partial \bar{M}^2}{\partial r}. \quad (13)$$

The system is stable (oscillatory) for $\partial \bar{M}^2 / \partial r > 0$, and unstable for $\partial \bar{M}^2 / \partial r < 0$.

As an example, consider a laboratory setup in which a viscous fluid is confined between two cylinders. Because the fluid is viscous, it sticks to both cylinders. If the inner cylinder is spinning while the outer one is not, the (squared) angular momentum of the fluid will decrease outward, so the flow will be unstable. If the outer cylinder is spinning while the inner one is not, the flow will be stable.

A geophysical perspective

Now we repeat the analysis from a geophysical perspective, using a rotating reference frame, which introduces Coriolis terms, and also including variations of the potential temperature with height. We use spherical coordinates $(\lambda, \varphi, \theta)$, where λ is longitude, φ is latitude, and θ is the potential temperature, which we assume to be conserved, i.e., we assume no heating.

To avoid unnecessary complications, we will also assume that there are no variations with longitude. This is justified because the physics of inertial instability relate to spatial variations of the angular momentum, which occur mainly with latitude. Inertial instability is sometimes called “symmetric instability” in the atmospheric science literature. This is not meant to suggest that inertial instability has to be independent of longitude. It could occur, for example, in the vicinity of a local maximum of the zonal wind, i.e., a “jet max.”

The Lagrangian time derivative can be written as

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \frac{v}{a} \frac{\partial}{\partial \varphi}, \quad (14)$$

where the partial derivatives are taken along θ surfaces. The equations of horizontal motion are given by

$$\frac{Du}{Dt} - \left(2\Omega + \frac{u}{a \cos \varphi} \right) v \sin \varphi = -\frac{1}{a \cos \varphi} \frac{\partial s}{\partial \lambda}, \quad (15)$$

$$\frac{Dv}{Dt} + \left(2\Omega + \frac{u}{a \cos \varphi} \right) u \sin \varphi + \Omega^2 a \cos \varphi \sin \varphi = -\frac{1}{a} \frac{\partial s}{\partial \varphi}, \quad (16)$$

where Ω is the magnitude of the angular velocity of the Earth’s rotation, a is the radius of the Earth, φ is latitude, and s is the dry static energy. The term $\Omega^2 a \cos \varphi \sin \varphi$ in (16) represents the centrifugal acceleration. Comparison of (16) with (3) shows that the u^2/r term of (3) appears with a minus sign, while the $u^2 \tan \varphi / a$ term of (16) does not. The reason is that the radial coordinate in (3) essentially points out away from the pole, so it is analogous to $a \left(\frac{\pi}{2} - \varphi \right)$, i.e., it decreases with latitude, whereas Eq. (16) uses latitude itself as the “radial” coordinate.

Starting from Eq. (15), we can show that

$$\frac{DM}{Dt} = 0, \quad (17)$$

where

$$M \equiv a \cos \varphi (u + \Omega a \cos \varphi) \quad (18)$$

is the component of the angular momentum, per unit mass, that points in the direction of the Earth's axis of rotation, and we have used

$$v \equiv a \frac{D\varphi}{Dt}. \quad (19)$$

Using (18), we can rewrite (16) as

$$\frac{Dv}{Dt} + \frac{M^2 \sin \varphi}{a^3 \cos^3 \varphi} = -\frac{1}{a} \frac{\partial s}{\partial \varphi}. \quad (20)$$

We now linearize the system about a basic state in which the flow is purely zonal and in gradient-wind balance, so that $\bar{v} = 0$ and $\frac{\bar{M}^2 \sin \varphi}{a^3 \cos^3 \varphi} = -\frac{1}{a} \frac{\partial \bar{s}}{\partial \varphi}$. The linearized version of (20) can be written as

$$\frac{\partial v'}{\partial t} + \left(\frac{2\bar{M} \sin \varphi}{a^3 \cos^3 \varphi} \right) M' = -\frac{1}{a} \frac{\partial s'}{\partial \varphi}, \quad (21)$$

and the linearized version of (17) is

$$\frac{\partial M'}{\partial t} + \frac{v'}{a} \frac{\partial \bar{M}}{\partial \varphi} = 0. \quad (22)$$

To investigate inertial stability and instability, we ignore the pressure-gradient term of (21), which plays only a secondary role. After substituting $(v', M') = (\hat{v}, \hat{M}) e^{\sigma t}$, our system reduces to

$$\sigma \hat{v} + \left(\frac{2\bar{M} \sin \varphi}{a^3 \cos^3 \varphi} \right) \hat{M} = 0, \quad (23)$$

$$\frac{\partial \bar{M}}{\partial \varphi} \frac{\hat{v}}{a} + \sigma \hat{M} = 0. \quad (24)$$

For nontrivial solutions, we need

$$\begin{aligned} \sigma^2 &= \left(\frac{2\bar{M} \sin \varphi}{a^3 \cos^3 \varphi} \right) \frac{1}{a} \frac{\partial \bar{M}}{\partial \varphi} \\ &= \frac{\sin \varphi}{a^4 \cos^3 \varphi} \frac{\partial \bar{M}^2}{\partial \varphi}. \end{aligned} \quad (25)$$

This result is consistent with (13), again taking into account that the radial coordinate in (13) is proportional to “minus” the latitudinal coordinate in (25). *In either hemisphere*, the system is *inertially stable if the angular momentum decreases towards the pole* along isentropic surfaces, and *inertially unstable if the angular momentum increases towards the pole* along isentropic surfaces. In a neutrally stable state, the angular momentum is constant along isentropic surfaces.

Rewriting the stability criterion in terms of vorticity

The criterion for inertial instability is often expressed in terms of vorticity rather than angular momentum. Here’s how that works. In spherical coordinates, the vertical component of the absolute vorticity is given by

$$\zeta + f = \frac{1}{a \cos \varphi} \frac{\partial v}{\partial \lambda} - \frac{1}{a \cos \varphi} \frac{\partial}{\partial \varphi} (u \cos \varphi) + 2\Omega \sin \varphi. \quad (26)$$

On the other hand, the meridional derivative of the angular momentum is

$$\frac{1}{a} \frac{\partial M}{\partial \varphi} = \frac{\partial}{\partial \varphi} (u \cos \varphi) - 2\Omega a \cos \varphi \sin \varphi. \quad (27)$$

Comparing (26) and (27), we see that for a purely zonal flow

$$\zeta + f = \frac{-1}{a \cos \varphi} \frac{\partial M}{\partial \varphi} .$$

(28)

This allows us to rewrite (25) as

$$\sigma^2 \equiv \frac{-2 \sin \varphi}{a^3 \cos^2 \varphi} \overline{M} (\overline{\zeta} + f) .$$

(29)

For the zonally averaged flow we can safely assume that $\overline{M} > 0$, except possibly close to the poles. It follows that the sign of σ^2 is determined by the sign of $-\sin \varphi (\overline{\zeta} + f)$. Inertial instability occurs when $\sigma^2 > 0$. In the Northern Hemisphere, where $\sin \varphi > 0$, the criterion for instability is satisfied for $\overline{\zeta} + f < 0$, and in the Southern Hemisphere it is satisfied for $\overline{\zeta} + f > 0$. In either hemisphere, inertial instability occurs when the absolute vorticity has “the wrong sign.” Since $\overline{\zeta} + f$ passes through zero near the Equator, inertial instability is relatively easy to excite there. The criterion for inertial instability can be satisfied when absolute vorticity is advected across the Equator (e.g., Thomas and Webster, 1997).

Conditional symmetric instability

Emanuel (1979, 1982) pointed out that when the air is saturated with water vapor the criterion for inertial instability can be expressed in terms of angular momentum variations along surfaces of constant equivalent potential temperature (or moist static energy), rather than surfaces of constant dry potential temperature. For saturated motion, instability can occur when the angular momentum decreases poleward along surfaces of constant saturation moist static energy. Another way of saying the same thing is that, for saturated motion, instability can occur when the saturation moist static energy decreases upward along surfaces of constant angular momentum.

When the zonal wind does not vary with height, surfaces of constant angular momentum are vertical. When the zonal wind varies with height, however, e.g., below a jet stream or in the vicinity of a front, angular momentum surfaces are tilted in the latitude-height plane. It is possible for the saturation moist static energy to decrease along such tilted surfaces of constant angular momentum even when it does not decrease upward. Emanuel (1983b) called this conditional symmetric instability, a term that is now in common use in the forecasting community. Emanuel argued that conditional symmetric instability is relevant to extratropical squall lines that form in regions of strong vertical wind shear.

References and Bibliography

- Emanuel, K. A., 1979: Inertial instability and mesoscale convective systems. Part I: Linear theory of inertial instability. *J. Atmos. Sci.*, **36**, 2425-2449.
- Emanuel, K. A., 1982: Inertial instability and mesoscale convective systems. Part II: Symmetric CISK in a baroclinic flow. *J. Atmos. Sci.*, **39**, 1080-1097.
- Emanuel, K. A., 1983a: The Lagrangian parcel dynamics of moist symmetric instability. *J. Atmos. Sci.*, **40**, 2368-2376.
- Emanuel, K. A., 1983b: On assessing local conditional symmetric instability from atmospheric soundings. *Mon. Wea. Rev.*, **111**, 2016-2033.
- Hack, J. J., W. H. Schubert, D. E. Stevens, and H.-C. Kuo, 1989: Response of the Hadley circulation to convective forcing in the ITCZ. *J. Atmos. Sci.*, **46**, 2957-2913.
- Holton, J. R., 2004: *An Introduction to Dynamic Meteorology* (4th Edition). Elsevier Academic Press, 535 pp.
- Sawyer, J. S., 1949: The significance of dynamic instability in atmospheric motions. *Quart. J. Roy. Meteor. Soc.*, **75**, 364-374.
- Solberg, P. H., 1936: Le mouvement d'inertie de l'atmosphère stable et son rôle dans la théorie des cyclones. *U.G.G.I. Assoc. Meteorol. Edinburgh.*, 66-82
- Strutt, J. W. ("Lord Rayleigh"), 1916: On the Dynamics of Revolving Fluids. *Proc. Roy. Soc. London*, **A93**, 447-453.
- Thomas, R. A., and P. J. Webster, 1997: The role of inertial instability in determining the location and strength of near-equatorial convection. *Quart. J. Roy. Meteor. Soc.*, **123**, 1445-1482.