Properties of the continuous equations

In this chapter, we consider a highly idealized version of the momentum equation: Shallow water, one dimension, no rotation. In a later chapter, we will go to two dimensions with rotation, and bring in the effects of vorticity, which are extremely important.

Consider the one-dimensional shallow-water equations, with bottom topography, without rotation and with $v = 0$. The prognostic variables are the water depth or mass, $h$, and the speed, $u$. The exact equations are

\[
\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(hu) = 0, \tag{1}
\]

and

\[
\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left[K + g(h + h_s)\right] = 0. \tag{2}
\]

Here

\[
K \equiv \frac{1}{2} u^2 \tag{3}
\]

is the kinetic energy per unit mass, $g$ is the acceleration of gravity, and $h_s$ is the height of the bottom topography. In Eq. (2), the vorticity has been assumed to vanish, which is reasonable in the absence of rotation and in one dimension. The effects of vorticity are of course absolutely critical in geophysical fluid dynamics; they will be discussed in a later chapter.

*The design of the scheme is determined by a sequence of choices.* We should welcome the opportunity to make the *best possible* choices. The first thing that we have to choose is the
particular form of the continuous equations that the space-differencing scheme is designed to mimic. Eq. (2) is one possible choice for the continuous form of the momentum equation. An alternative choice is

$$\frac{\partial}{\partial t}(hu) + \frac{\partial}{\partial x}(huv) + gh \frac{\partial}{\partial x}(h + hs) = 0,$$

(4)
i.e., the flux form of the momentum equation, which can be derived by combining (1) and (2).

The continuous shallow-water equations have important “integral properties,” which we will use as a guide in the design of our space-differencing scheme. For example, if we integrate (1) with respect to $x$, over a closed or periodic domain, we obtain

$$\frac{d}{dt} \left( \int_{\text{domain}} h \, dx \right) = 0,$$

(5)
which means that mass is conserved.

Using

$$h \frac{\partial h}{\partial x} = \frac{\partial}{\partial x} \left( \frac{h^2}{2} \right),$$

(6)
we can rewrite (4) as

$$\frac{\partial}{\partial t}(hu) + \frac{\partial}{\partial x} \left( huv + \frac{gh^2}{2} \right) = -gh \frac{\partial hs}{\partial x}.$$

(7)
The momentum per unit area is $hu$. If we integrate with respect to $x$, over a periodic domain, we obtain

$$\frac{d}{dt} \left( \int_{\text{domain}} hu \, dx \right) = - \int_{\text{domain}} gh \frac{\partial hs}{\partial x} \, dx.$$

(8)
This shows that in the absence of topography, i.e., if $\frac{\partial hs}{\partial x} = 0$ everywhere, the domain average of $hu$ is invariant, i.e., momentum is conserved. When $hs$ is spatially variable, the atmosphere and the “solid earth” can exchange momentum through the pressure field.
The flux form of the kinetic energy equation can be derived by multiplying (1) by $K$ and (2) by $hu$, and adding the results, to obtain

$$\frac{\partial}{\partial t}(hK) + \frac{\partial}{\partial x}(huK) + hu \frac{\partial}{\partial x} \left[ g(h + h_s) \right] = 0. \quad (9)$$

The last term of (9) represents conversion between potential and kinetic energy.

The potential energy equation can be derived by multiplying (1) by $g(h + h_s)$ to obtain

$$\frac{\partial}{\partial t} \left[ hg \left( h_s + \frac{1}{2}h \right) \right] + g(h + h_s) \frac{\partial}{\partial x} (hu) = 0, \quad (10)$$

or

$$\frac{\partial}{\partial t} \left[ hg \left( h_s + \frac{1}{2}h \right) \right] + \frac{\partial}{\partial x} \left[ hu h + h_s \right] - hu \frac{\partial}{\partial x} \left[ g(h + h_s) \right] = 0. \quad (11)$$

Here $g \left( h_s + \frac{1}{2}h \right)$ can be interpreted as the potential energy per unit mass of a particle that is “half-way up” in the water column. The middle term represents both advection of potential energy and the horizontal redistribution of energy by pressure-work. The last term represents conversion between kinetic and potential energy; compare with (9). In deriving (10), we have assumed that $h_s$ is independent of time. This assumption can easily be relaxed, at the cost of an additional term in (11).

When we add (9) and (10), the energy conversion terms cancel, and we obtain a statement of the conservation of total energy, i.e.,

$$\frac{\partial}{\partial t} \left\{ h \left[ K + g \left( h_s + \frac{1}{2}h \right) \right] \right\} + \frac{\partial}{\partial x} \left\{ hu \left[ K + g(h + h_s) \right] \right\} = 0. \quad (12)$$

The integral of (12) over a closed or periodic domain gives

$$\frac{d}{dt} \int_{\text{domain}} h \left[ K + g \left( h_s + \frac{1}{2}h \right) \right] dx = 0,$$

which shows that the domain-integrated total energy is conserved.
The spatially discrete case

Now consider finite-difference approximations to (1) and (2). We keep the time derivatives continuous, and explore the effects of space differencing only. *We use a staggered grid, with \( h \) defined at integer points (hereafter called mass points) and \( u \) at half-integer points (hereafter called wind points).* This can be viewed as a one-dimensional version of the C grid. The grid spacing, \( \Delta x \), is assumed to be uniform. Our selection of this particular grid is a second choice made in the design of the space-differencing scheme.

The finite difference version of the continuity equation is

\[
\frac{dh_i}{dt} + \left( \frac{hu_{i+\frac{1}{2}} - hu_{i-\frac{1}{2}}}{\Delta x} \right) = 0. \tag{14}
\]

It should be understood that

\[
h_{i+\frac{1}{2}}u_{i+\frac{1}{2}} \equiv (hu)_{i+\frac{1}{2}}. \tag{15}
\]

The “wind-point” masses, e.g., \( h_{i+\frac{1}{2}} \), are undefined at this stage. The finite-difference approximation used in (14) is consistent with second-order accuracy in space, although we cannot really determine the order of accuracy until the finite-difference form of the mass flux has been specified. We have already discussed how the “flux form” used in (14) makes it possible for the model to conserve mass, i.e.,

\[
\frac{d}{dt} \left( \sum_{\text{domain}} h_i \right) = 0, \tag{16}
\]

and this is true regardless of how \( h_{i+\frac{1}{2}} \) is defined. Eq. (16) is analogous to (5).

A finite-difference momentum equation that is modeled after (2) is

\[
\frac{du_{i+\frac{1}{2}}}{dt} + \left( \frac{K_{i+1} - K_i}{\Delta x} \right) + g \left[ \frac{(h + h_3)_{i+1} - (h + h_3)_i}{\Delta x} \right] = 0. \tag{17}
\]

The kinetic energy per unit mass, \( K_i \), is undefined at this stage, but resides at mass points. The finite-difference approximations used in (17) are consistent with second-order accuracy in space, although we cannot really determine the order of accuracy until the finite-difference forms of the
mass flux and kinetic energy are specified. Although Eq. (17) does not have a “flux form,” we might (or might not) be able to use the continuity equation to show that it is consistent with (i.e., can be derived from) a flux form. This is discussed further below.

Multiply (17) by $h_{i+\frac{1}{2}}$ to obtain

$$h_{i+\frac{1}{2}} \frac{du_{i+\frac{1}{2}}}{dt} + h_{i+\frac{1}{2}} \left( \frac{K_{i+1} - K_i}{\Delta x} \right) + gh_{i+\frac{1}{2}} \left[ \frac{(h+h_s)_{i+1} - (h+h_s)_i}{\Delta x} \right] = 0.$$  
(18)

In order to mimic the differential relationship (6), we must require that

$$h_{i+\frac{1}{2}} \left( \frac{h_{i+1} - h_i}{\Delta x} \right) = \left( \frac{h_{i+1}^2 - h_i^2}{2 \Delta x} \right),$$  
(19)

which can by satisfied by choosing

$$h_{i+\frac{1}{2}} = \frac{h_{i+1} + h_i}{2}.$$  
(20)

This choice is required for momentum conservation. In view of (20), we can write

$$(hu)_{i+\frac{1}{2}} = \left( \frac{h_{i+1} + h_i}{2} \right) u_{i+\frac{1}{2}}.$$  
(21)

Combining (20) with the continuity equation (14), we see that we can write a continuity equation for the wind points, as follows:

$$\frac{dh_{i+\frac{1}{2}}}{dt} + \frac{1}{2 \Delta x} \left[ (hu)_{i+\frac{1}{2}} - (hu)_{i-\frac{1}{2}} \right] = 0.$$  
(22)

It should be clear from the form of (22) that the “wind-point mass” is actually conserved by the model. Of course, we do not actually use (22) when we integrate the model; instead we use (14). Nevertheless, (22) will be satisfied, because it can be derived from (14) and (20). An alternative form of (22) is
\[ \frac{dh}{dt} + \frac{1}{\Delta x} \left[ (hu)_{i+1} - (hu)_i \right] = 0, \]

where

\[ (hu)_{i+1} \equiv \frac{1}{2} \left[ (hu)_{i+\frac{3}{2}} + (hu)_{i+\frac{1}{2}} \right] \quad \text{and} \quad (hu)_i \equiv \frac{1}{2} \left[ (hu)_{i+\frac{1}{2}} + (hu)_{i-\frac{1}{2}} \right]. \]

(23)

Now add (18) and \( u_{i+\frac{1}{2}} \) times (23), and use (20), to obtain what “should be” the flux form of the momentum equation, analogous to (4):

\[ \frac{d}{dt} \left( h_{i+\frac{1}{2}} u_{i+\frac{1}{2}} \right) + h_{i+\frac{1}{2}} \left( K_{i+1} - K_i \right) + u_{i+\frac{1}{2}} \frac{1}{\Delta x} \left[ (hu)_{i+1} - (hu)_i \right] + g \left( \frac{h^2_{i+1} - h^2_i}{2\Delta x} \right) = -gh_{i+\frac{1}{2}} \left[ (h_s)_{i+1} - (h_s)_i \right]. \]

(25)

Is this really a flux form, or not? The answer is, it depends on how we define \( K_i \). Suppose that \( K_i \) is defined by

\[ K_i \equiv \frac{1}{2} u_{i+\frac{1}{2}} u_{i-\frac{1}{2}}. \]

(26)

Other possible definitions of \( K_i \) will be discussed later. Using (26) and (24), we can write

\[ h_{i+\frac{1}{2}} \left( K_{i+1} - K_i \right) + u_{i+\frac{1}{2}} \frac{1}{\Delta x} \left[ (hu)_{i+1} - (hu)_i \right] = \frac{1}{2\Delta x} \left[ \left( \frac{h_{i+\frac{3}{2}} + h_{i+\frac{1}{2}}}{2} \right) u_{i+\frac{1}{2}} - \left( \frac{h_{i+\frac{1}{2}}}{2} \right) u_{i+\frac{1}{2}} \right] + \left( \frac{h_{i+\frac{3}{2}} + h_{i+\frac{1}{2}}}{2} \right) \left( \frac{h_{i+\frac{1}{2}} + h_{i+\frac{1}{2}}}{2} \right) \left( \frac{h_{i+\frac{1}{2}} + h_{i+\frac{1}{2}}}{2} \right) \left( \frac{h_{i+\frac{1}{2}} - h_{i+\frac{1}{2}}}{2} \right) u_{i+\frac{1}{2}}. \]

(27)
This is a flux form. The momentum flux at the point \( i \) is 
\[
\left( \frac{h_{i+\frac{1}{2}} + h_{i-\frac{1}{2}}}{2} \right) u_{i-\frac{1}{2}} u_{i+\frac{1}{2}} \text{, and the}
\]
momentum flux at the point \( i+1 \) is 
\[
\left( \frac{h_{i+\frac{3}{2}} + h_{i-\frac{3}{2}}}{2} \right) u_{i+\frac{3}{2}} u_{i+\frac{1}{2}} \text{. Because (27) is a flux form,}
\]
momentum will be conserved by the scheme if we define the kinetic energy by (26).

Note, however, that there are two problems with (26). When \( u \) is dominated by the \( 2\Delta x \)-mode, (26) will give a negative value of \( K_{i} \), which is unphysical. In addition, when \( u \) is dominated by the \( 2\Delta x \)-mode, the momentum flux will always be negative for the \( 2\Delta x \)-mode, i.e., momentum will always be transferred in the \(-x\) direction, assuming that the interpolated masses that appear in the momentum fluxes are positive.

Next, we derive the kinetic energy equation. For this discussion, we return to general form of \( K_{i} \); Eq. (26) will not be used. Recall that the kinetic energy is defined at mass points. To begin the derivation, multiply (17) by \((hu)_{i+\frac{1}{2}}\) to obtain
\[
(hu)_{i+\frac{1}{2}} \frac{du}{dt} + (hu)_{i+\frac{1}{2}} \left( \frac{K_{i+1} - K_{i}}{\Delta x} \right) + g(hu)_{i+\frac{1}{2}} \left[ \frac{(h + h_{s})_{i+1} - (h + h_{s})_{i}}{\Delta x} \right] = 0 \tag{28}
\]
Rewrite (28) for grid point \( i - \frac{1}{2} \), simply by subtracting one from each subscript:
\[
(hu)_{i-\frac{1}{2}} \frac{du}{dt} + (hu)_{i-\frac{1}{2}} \left( \frac{K_{i} - K_{i+1}}{\Delta x} \right) + g(hu)_{i-\frac{1}{2}} \left[ \frac{(h + h_{s})_{i} - (h + h_{s})_{i+1}}{\Delta x} \right] = 0 \tag{29}
\]
Now add (28) and (29), and multiply the result by \( \frac{1}{2} \) to obtain the arithmetic mean:
\[
\frac{1}{2} \left[ (hu)_{i+\frac{1}{2}} \frac{du}{dt} + (hu)_{i-\frac{1}{2}} \frac{du}{dt} \right] + \frac{1}{2} \left[ (hu)_{i+\frac{1}{2}} \left( \frac{K_{i+1} - K_i}{\Delta x} \right) + (hu)_{i-\frac{1}{2}} \left( \frac{K_i - K_{i-1}}{\Delta x} \right) \right] + \frac{g}{2} \left[ (hu)_{i+\frac{1}{2}} \left( \frac{(h+h_S)_{i+1} - (h+h_S)_i}{\Delta x} \right) + (hu)_{i-\frac{1}{2}} \left( \frac{(h+h_S)_i - (h+h_S)_{i-1}}{\Delta x} \right) \right] = 0. 
\]

(30)

This is an advective form of the kinetic energy equation.

Next we try to derive, from (30) and (14), a flux form of the kinetic energy equation. Begin by multiplying (14) by \( K_i \):

\[
K_i \left\{ \frac{dh_i}{dt} + \left[ (hu)_{i+\frac{1}{2}} - \frac{(hu)_{i-\frac{1}{2}}}{\Delta x} \right] \right\} = 0. 
\]

(31)

Keep in mind that we still do not know what \( K_i \) is; we have just multiplied the continuity equation by a mystery variable. Add (31) and (30) to obtain

\[
K_i \frac{dh_i}{dt} + \frac{1}{2} \left[ (hu)_{i+\frac{1}{2}} \frac{du}{dt} + (hu)_{i-\frac{1}{2}} \frac{du}{dt} \right] + \frac{1}{\Delta x} \left[ K_i + \frac{1}{2} (K_{i+1} - K_i) - \frac{1}{2} (K_i - K_{i-1}) \right] + \frac{g}{2} \left[ (hu)_{i+\frac{1}{2}} \left( \frac{(h+h_S)_{i+1} - (h+h_S)_i}{\Delta x} \right) + (hu)_{i-\frac{1}{2}} \left( \frac{(h+h_S)_i - (h+h_S)_{i-1}}{\Delta x} \right) \right] = 0. 
\]

(32)

Eq. (32) “should be” a flux form of the kinetic energy equation. Is it really? To answer this question, we analyze the various terms of (32) one by one.

The advection terms on the second line of (32) are very easy to deal with. They can be rearranged to

\[
\frac{1}{\Delta x} \left[ (hu)_{i+\frac{1}{2}} \left( \frac{1}{2} (K_{i+1} + K_i) - \frac{1}{2} (K_i + K_{i-1}) \right) \right]. 
\]

(33)
This has the form of a “finite-difference flux divergence.” The conclusion is that these terms are consistent with kinetic energy conservation under advection, simply by virtue of their form, regardless of the method chosen to determine $K_i$.

Next, consider the energy conversion terms on the third line of (32), i.e.,

$$
\frac{g}{2} \left\{ (hu)_{i+\frac{1}{2}} \left[ \frac{(h+h_S)_{i+1}-(h+h_S)_i}{\Delta x} \right] + (hu)_{i-\frac{1}{2}} \left[ \frac{(h+h_S)_i-(h+h_S)_{i-1}}{\Delta x} \right] \right\},
$$

(34)

We want to compare these terms with the corresponding terms of the finite-difference form of the potential energy equation, which can be derived by multiplying (14) by $g(h+h_S)_i$:

$$
d dt \left\{ h_i g \left( h_i + \frac{1}{2} h \right) \right\} + g(h+h_S)_i \left[ \frac{(hu)_{i+\frac{1}{2}}-(hu)_{i-\frac{1}{2}}}{\Delta x} \right] = 0.
$$

(35)

Eq. (35) is analogous to (10). We want to recast (35) so that we see advection of potential energy, as well as the energy conversion term corresponding to (34); compare with (11). To accomplish this, write

$$
\frac{d}{dt} \left[ h_i g \left( h_i + \frac{1}{2} h \right) \right] + ADV_i - \frac{g}{2} \left\{ (hu)_{i+\frac{1}{2}} \left[ \frac{(h+h_S)_{i+1}-(h+h_S)_i}{\Delta x} \right] + (hu)_{i-\frac{1}{2}} \left[ \frac{(h+h_S)_i-(h+h_S)_{i-1}}{\Delta x} \right] \right\} = 0.
$$

(36)

where “$ADV_i$” represents the advection of potential energy, in flux form. At this point, we do not have an expression for $ADV_i$, but we will let the equations tell us how to compute it. The second line of (36) is a copy of the energy conversion terms of (32), but with the sign reversed. We require that (36) be equivalent to (35), and ask what form of $ADV_i$ is implied by this requirement. The answer is:

$$
ADV_i = \frac{g}{2\Delta x} \left\{ (hu)_{i+\frac{1}{2}} \left[ (h+h_S)_{i+1} + (h+h_S)_i \right] - (hu)_{i-\frac{1}{2}} \left[ (h+h_S)_i + (h+h_S)_{i-1} \right] \right\}.
$$

(37)

This does indeed have the form of a finite-difference flux divergence, as desired.
What we have shown up to this point is that the conservation of potential energy and the cancellation of the energy conversion terms have both turned out to be pretty easy. The form of $K_i$ is still up to us.

We are not quite finished, however, because we have not yet examined the time-rate-of-change terms of (32). Obviously, the first line of (32) must be analogous to $\frac{\partial}{\partial t}(hK)$. For convenience, we define

$$
(KE \text{ tendency})_i \equiv K_i \frac{dh_i}{dt} + \frac{1}{2} \left[ (hu)_{i+\frac{1}{2}} \frac{d}{dt} u_{i+\frac{1}{2}} + (hu)_{i-\frac{1}{2}} \frac{d}{dt} u_{i-\frac{1}{2}} \right].
$$

(38)

Substituting for the mass fluxes from (21), we can write (38) as

$$
(KE \text{ tendency})_i \equiv K_i \frac{dh_i}{dt} + \frac{1}{8} \left[ (h_{i+1} + h_i) \frac{d}{dt} u_{i+\frac{1}{2}} + (h_i + h_{i-1}) \frac{d}{dt} u_{i-\frac{1}{2}} \right].
$$

(39)

The requirement for kinetic energy conservation is

$$
\sum_{\text{domain}} (KE \text{ tendency})_i = \sum_{\text{domain}} \frac{d}{dt}(h_iK_i).
$$

(40)

Note that only the sums over $i$ must agree; it is not necessary that

$$
K_i \frac{dh_i}{dt} + \frac{1}{8} \left[ (h_{i+1} + h_i) \frac{d}{dt} u_{i+\frac{1}{2}} + (h_i + h_{i-1}) \frac{d}{dt} u_{i-\frac{1}{2}} \right] \text{ be equal to } \frac{d}{dt}(h_iK_i) \text{ for each } i.
$$

To complete our check of kinetic energy conservation, we substitute for $K_i$ on the right-hand side of (40), and check to see whether the resulting equation is actually satisfied.

The bad news is that, if we use (26), Eq. (40) is not satisfied. This means that we cannot have both momentum conservation under advection and kinetic energy conservation, when we start from the continuous form of (2). On the other hand, we did not like (26) anyway, because for the $2\Delta x$ wave it gives negative kinetic energy and a momentum flux in the negative $x$ direction.

The good news is that there are ways to satisfy (40). Two alternative definitions of the kinetic energy are
\[ K_i \equiv \frac{1}{4} \left( u_{i+\frac{1}{2}}^2 + u_{i-\frac{1}{2}}^2 \right), \]

(41)

and

\[ h_i K_i \equiv \frac{1}{4} \left( h_{i+\frac{1}{2}} u_{i+\frac{1}{2}}^2 + h_{i-\frac{1}{2}} u_{i-\frac{1}{2}}^2 \right). \]

(42)

With either of these definitions, \( K_i \) cannot be negative. We can show that the sum over the domain of \( h_i K_i \) given by (41) is equal to the sum over the domain of \( h_i K_i \) given by (42). Either choice allows (40) to be satisfied, so both are consistent with kinetic energy conservation under advection. On the other hand, neither is consistent with momentum conservation under advection.

To sum up: When we start from the continuous form of (2), we can have either momentum conservation under advection or kinetic energy conservation under advection, but not both. Which is better depends on the application.

An alternative approach is to start from a finite-difference form of the momentum equation that mimics (4). In that case, we can conserve both momentum under advection and kinetic energy under advection. You are asked to demonstrate this in Problem 2 at the end of this chapter.

When we generalize to the two-dimensional shallow-water equations with rotation, there are very important additional considerations having to do with vorticity, and the issues discussed here have to be revisited. This is discussed in Chapter 11.

Summary

We have explored the conservation properties of spatial finite-difference approximations of the momentum and continuity equations for one-dimensional non-rotating flow, using a staggered grid. We were able to find a scheme that guarantees conservation of mass, conservation of momentum in the absence of bottom topography, conservation of kinetic energy under advection, conservation of potential energy under advection, and conservation of total energy in the presence of energy conversion terms.

This chapter has introduced several new things. This is the first time that we have considered the advection terms of the momentum equation. This is the first time that we have discussed energy conversions and total energy conservation. In addition, the chapter illustrates a way of thinking about the trade-offs that must be weighed in the design of a scheme, as various alternative choices each have advantages and disadvantages.

An Introduction to Numerical Modeling of the Atmosphere
Problems

1. Show that if we use (26) it is not possible to conserve kinetic energy under advection.

2. Starting from a finite-difference form that mimics (4), show that it is possible to conserve both momentum and total energy. Use the C grid, and keep the time derivatives continuous.