III. Prototype instabilities

A. Introduction

Broadly speaking, there are two sources from which a growing atmospheric disturbance can draw its kinetic energy: the kinetic and available potential energies of the mean flow. The first type of instability is sometimes called "mechanical," while the second is called "buoyant" or "convective." We deal with both types in this course.

The two varieties of mechanical instability which we consider are the inflectional or Rayleigh instability, which is always associated with a maximum in the vorticity distribution, and the inertial instability, which can only occur in a rotating fluid, and which results from an unstable distribution of angular momentum. Inflectional instability is certainly responsible for the maintenance of turbulence in the stable PRE, and may also play an important role in some other boundary layer phenomena, such as cloud streets and billows.
instability is probably less important in the PBL, but may sometimes help to organize squall lines.

Convective instabilities play a prominent role in this course, most notably as manifested in cumulus convection, but also in maintaining the convective turbulence of the deep, unstable PBLs. The growth rates and characteristic shapes of convective disturbances are strongly influenced by shear in the atmosphere, and may occasionally be significantly modified by rotation.

But the study of these prototype instabilities can't bring us very close to the real world of atmospheric turbulence, because analytical results are, for the most part, valid only for motions of infinitesimal amplitude, while the real motions are of distinctly finite amplitude. A cumulus circulation, for example, quickly amplifies and becomes subject to inflectional instabilities at its own boundary. And an essential control on the intensities of all these instabilities is their tendency to relieve the conditions which led to their growth.
by stabilizing the basic state.

For these reasons, we investigate only the essential mechanism of each instability, and provide, wherever possible, empirical evidence of the finite-amplitude effects which the linear theories can never explain. References to more elaborate theoretical discussions are cited, as appropriate.
B. Inflectional instability

The simplest model of inflectional instability is based on consideration of a vortex sheet, as pictured in Fig. 3.1. Part (a) of the figure depicts a stable configuration, in which the vorticity increases monotonically from the lower to the upper part of the figure. A vortex displaced, as indicated, from its equilibrium position will tend to be carried back by the circulations induced by the neighboring vortices. But if the vorticity has a local maximum, as in part (b) of the figure, then a displaced vortex tends to be carried further away. Such a vorticity distribution is therefore unstable.

The following, more quantitative analysis is based on a portion of Chapter XI of Chandrasekhar (1961). Consider a flow along the x-axis, sheared in the z-direction. We adopt the Boussinesq system of equations (Appendix A), which have the following form:
Stable configuration
(no vorticity maximum)

(a)

Unstable configuration
(vorticity maximum)

(b)

Fig. 2.1
\[ s_0 \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + \omega \frac{\partial u}{\partial z} \right) = -\frac{\partial p}{\partial x} \quad (1) \]

\[ s_0 \left( \frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} \right) = -\frac{\partial p}{\partial y} \quad (2) \]

\[ s_0 \left( \frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} \right) = -\frac{\partial p}{\partial z} - \frac{\partial g}{\partial y} \quad (3) \]

\[ \frac{\partial p}{\partial t} + u \frac{\partial p}{\partial x} + \omega \frac{\partial p}{\partial z} = 0 \quad (4) \]

\[ \frac{\partial u}{\partial x} + \frac{\partial w}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad (5) \]

We seek wave-like solutions of the form

\[ (\cdot) = (\cdot) \exp [i(kx + ly + \omega t)] \quad (6) \]
Substituting (a) into (1-5), we obtain

\[ e^{-ikL} \left( i \frac{dU}{dz} \right) E = -ikp \]  

(7)

\[ e^{-ikL} \left( (\sigma + kL) \right) \psi = -ikp \]  

(8)

\[ e^{-ikL} \left( (\sigma + kL) \right) \omega = - \frac{dp}{dz} - g \phi \]  

(9)

\[ (\sigma + kL) \psi + \omega \frac{dp}{dz} = 0 \]  

(10)

\[ i(ku + 2\omega) + \frac{d\omega}{dz} = 0 \]  

(11)

Forming the divergence equation, and using continuity, we get

\[ \int e^{-ikL} \left( (\sigma + kL) \frac{d\omega}{dz} - k \frac{dL}{dz} \omega \right) = -(k^2 + \ell^2) \psi \]  

(12)

and (9 - 10) give

\[ \int e^{-ikL} \left( (\sigma + kL) \omega = - (\sigma + kL) \frac{dp}{dz} - i \omega \frac{dp}{dz} \right) \]  

(13)
Elimination of $p$ between these gives

$$
\frac{d}{dz} \left[ \rho_0 (\sigma + kU) \frac{dw}{dz} - \rho_0 k \omega_0 \frac{dU}{dz} \right] - (k^2 + \sigma^2) \rho_0 (\sigma + kU) \omega = \rho_0 (k^2 + \sigma^2) \frac{d\rho_0}{dz} \left( \frac{\omega}{\sigma + kU} \right)
$$

(14)

We'll consider unbounded domains in which the motions must die out far from the shear zone:

$$w \to 0 \text{ as } z \to \pm \infty$$

(15)

Let $z = z_s (x, y, t)$ be a free surface. We now derive the form of the kinematic boundary condition at this surface. In order to do so, we must go back to the full continuity equation, which is

$$
\frac{\partial \nu}{\partial t} + \nabla \cdot (\nu \rho \mathbf{U}) = 0.
$$

(16)
Integrating (16) across the layer 
\( z_s - \varepsilon \leq z \leq z_s + \varepsilon \) (Fig. 3.2),

\[ z = z_s + \varepsilon \]

\[ z = z_s (x, y, t) \]

\[ z = z_s - \varepsilon \]

we obtain

\[
\left[ \int_{z_s - \varepsilon}^{z_s + \varepsilon} \left( \frac{\partial}{\partial t} \left( \rho v \right) + \nabla \cdot \left( \rho v u \right) \right) \right] - \int_{s_s}^{s_s + \varepsilon} \left[ \frac{\partial}{\partial t} \left( \rho v z_s + \nabla \cdot (\rho v z_s) - w_{s_s} \right) \right] \]

- \int_{s_s}^{s_s - \varepsilon} \left[ \frac{1}{\partial t} \left( \rho v z_s - \nabla \cdot (\rho v z_s) - w_{s_s} \right) \right] \]

= 0

(17)

As \( \varepsilon \to 0 \), the integrals in \( [ ] \) go to zero, and their derivatives also go to zero, so the terms drop out. We're left with
\[ \Delta \left( \int \left( \frac{\partial z^+}{\partial t} + u \cdot \nabla z^+ - \omega z^+ \right) \right) = 0 \]

(19)

Here \( \Delta(\cdot) \equiv (\cdot)_{s^+} - (\cdot)_{s^-} \).

we can write

\[ \int_{s^+} \left( \frac{\partial z^+}{\partial t} + u_{s^+} \cdot \nabla z^+ - \omega_{s^+} \right) = F \]

(20)

and

\[ \int_{s^-} \left( \frac{\partial z^-}{\partial t} + u_{s^-} \cdot \nabla z^- - \omega_{s^-} \right) = F \]

(21)

Note that (20-21) automatically satisfy (19). Here \( F \) is the mass flux across \( z_s(x,y,t) \). If \( z_s \) is a free surface, then \( F = 0 \), so that

\[ \frac{\partial z^+}{\partial t} + u_{s^+} \cdot \nabla z^+ - \omega_{s^+} = 0 \]

(22)

and

\[ \frac{\partial z^-}{\partial t} + u_{s^-} \cdot \nabla z^- - \omega_{s^-} = 0 \]

(23)

It follows that

\[ \Delta \left( u \cdot \nabla z_s - \omega \right) = 0 \]

(24)
Notice that (19) says that the normal component of the mass flux must be continuous, while (24) says that the normal component of the velocity must be continuous. The latter is true only at a free surface.

We now return to our earlier analysis. Substitution of (6) into (22) and (23) gives

\[ i(\sigma + kU)z_{s+} - w_{s+} = 0, \tag{25} \]

and

\[ i(\sigma + kU)z_{s-} - w_{s-} = 0, \tag{26} \]

respectively. Requiring that the fluid not tear, we obtain

\[ \Delta \left( \frac{w}{\sigma + kU} \right) = 0. \tag{27} \]

where \( \Delta(\cdot) = (\cdot)_{s+} - (\cdot)_{s-} \). Similarly, from (14) we find that

\[ \Delta \left\{ g \left( \frac{(\sigma + kU)\partial w}{\partial z} - kw \partial \frac{U}{\partial z} \right) \right\} = g(k^2 + \lambda^2) \Delta \phi \left( \frac{w}{\sigma + kU} \right). \tag{28} \]
Now suppose that $U$ and $f_0$ are discontinuous at the free surface $z_s$. Taking $U$ and $f_0$ to be uniform except at the interface, we find that (14) can be simplified to

\[
\left[ \frac{\partial^2}{\partial z^2} - (k^2 + l^2) \right] u = 0 \quad \text{(29)}
\]

In view of the external boundary conditions given by (17), the solution is

\[
u_1 = A (\sigma + k u_1) \cdot \exp \left( -\sqrt{k^2 + l^2} \right) \quad (z < 0)
\]

\[
u_2 = A (\sigma + k u_2) \cdot \exp \left( -\sqrt{k^2 + l^2} \right) \quad (z > 0)
\]

Substitution of (30) into (28) gives

\[
\sigma^2 + 2k (x_1 u_1 + x_2 u_2) \sigma + k^2 (x_1 u_1^2 + x_2 u_2^2)
\]

\[
= g \sqrt{k^2 + l^2} (x_1 - x_2),
\]

\text{(31)}
where

\[ \alpha_i = \frac{f_i}{f_1 + f_2}, \quad i = 1, 2. \] (32)

Solving (31) as a quadratic equation for \( r \), we obtain

\[ r = \frac{-k(x_1w_1 + x_2w_2) \pm \sqrt{[g \sqrt{k^2 + l^2} (x_1 - x_2) + k^2 x_1 x_2 (w_1 - w_2)^2]}^{1/2}}{x_1 x_2 (w_1 - w_2)^2}. \] (33)

Notice that for \( k = 0 \), \( w_1 \), and \( w_2 \) drop out of the problem, and we just get a gravity wave. So all growing disturbances have \( k > 0 \). In fact, instability occurs for

\[ \sqrt{k^2 + l^2} > \frac{g(x_1 - x_2)}{x_1 x_2 (w_1 - w_2)^2 \cos^2 \theta}, \] (34)

where \( \theta \) is the angle between the wave number vector and the x-direction. The minimum wave number occurs for \( \cos^2 \theta = 1 \). No matter how small \( |w_1 - w_2| \) is made, instability will still occur.
We now turn to the case of continuous $W$ and $p_0$. First, consider the following simple analysis. Exchange two parcels between heights $z$ and $z+dz$. The work done is $-g \rho dz$, per unit volume. The decrease in the kinetic energy which occurs as a result of this mixing, will be

$$\int \left[ \frac{1}{2} W^2 + \frac{1}{2} (W+dw)^2 - \frac{1}{2} \cdot \frac{1}{2} (W+U+du)^2 \right] = \frac{1}{4} \rho (du)^2$$

(35)

For instability, the decrease in kinetic energy must exceed the work done. This condition can be written as

$$R_i < \frac{1}{4},$$

(36)

where

$$R_i = -\frac{g}{\int \left( \frac{dW}{dz} \right)^2}$$

(37)

is the Richardson number. This argument does not show that (36) is a sufficient condition for instability — only that it is necessary. We now consider the problem more carefully.
Let
\[ f = f_0 e^{-\beta z}, \]  
and
\[ U = U_0 + \tanh \left( \frac{z}{d} \right), \]

where \( \beta \) and \( d \) are constant. Then
\[ (Ri) = q^2 \beta d^2 / U_0 \]
is a representative value of \( Ri \) in the vicinity of \( z = 0 \).

Taking \( \lambda = 0 \), for simplicity, we expand (14)
\[ \frac{\partial}{\partial z} \left( f^2 - k \frac{d^2 U}{d z^2} \right) \frac{\partial U}{\partial z} - \frac{q^2 k^2}{f_0} \frac{\partial \omega}{\partial f_0} \left( \frac{\omega}{\delta + kU} \right) \]
\[ + \int \frac{df_0}{\delta f_0} \left( \delta + kU \right) \frac{\partial \omega}{\partial z} - k \frac{dU}{dz} \frac{\partial U}{\partial z} = 0 \]

(41)

The second last term of (41), involving \( f_0 \), is in geophysical applications much larger than the last. This means that \( df_0 / dz \) matters for its effect on potential energy, but not for its effect on momentum.
Then (41) can be simplified to

\[
\frac{(T + U)}{k}\left(\frac{d^2}{dz^2} - k^2\right)\omega - \frac{d^2 U}{dz^2} \omega - \frac{2\alpha}{\eta_0}\frac{\omega}{\eta/k + \nu} = 0
\]

(42)

Nondimensionalize by scaling velocities with \( U_{oo} \) and lengths with \( d_j \), and define a nondimensional phase speed

\[
C = \frac{\omega}{kU_{oo}}
\]

(43)

Then (42) becomes

\[
(U - C)\left(\frac{d^2}{dz^2} - k^2\right)\omega - \frac{d^2 U}{dz^2} \omega + (Re)_{\infty} \left(\frac{\omega}{U - C}\right) = 0
\]

(44)

where all variables are now nondimensional \((U = \tanh z)\). We apply the boundary conditions

\[
\omega = 0 \text{ as } z \to \pm \infty.
\]

(45)

Instability occurs for complex values of \( C \). We assume that in the "marginal" (i.e., neutrally stable) state, any motions are
stationary \( (\sigma = \zeta = 0) \). Then to find the
marginal state we must solve

\[
L \left( \frac{\partial^2}{\partial z^2} - k^2 \right) \omega - \frac{d^2 L}{dz^2} \omega + (\text{Ri})_0 \frac{\omega}{L} = 0.
\]

(46)

Since \( L \) is a single-valued function of \( z \),
we can use \( L \) instead of \( z \) as our
vertical coordinate. We then obtain

\[
\frac{d}{dU} \left[ (1 - U^2) \frac{d\omega}{dU} \right] + \left[ 2 - \frac{k^2}{1 - U^2} + \frac{(\text{Ri})_0}{U^2(1 - U^2)} \right] \omega = 0,
\]

(47)

subject to the boundary conditions

\[
\omega = 0 \text{ at } U = \pm 1.
\]

(48)

We state without proof that the solution is

\[
\omega = \chi L \mu \left( 1 - U^2 \right)^\nu.
\]

(49)

where \( \chi \) is constant,

\[
\mu = \frac{1}{2} \left[ 1 + \sqrt{1 - 4(\text{Ri})_0^2} \right],
\]

(50)
and

$$\nu = \frac{1}{2} \sqrt{k^2 - (Ri)_0^2}. \quad (51)$$

Substitution of (49-51) into (47) shows that

$$(2\nu + \mu + 2) \cdot (2\nu + \mu + 1) = 0 \quad (52)$$

is the dispersion relation which relates $k$ to $(Ri)_0$. The first factor of (52) cannot be zero [as can be shown by substitution from (50) and (51)], so the second must vanish. Substitution for $\nu$ and $\mu$ shows that this is equivalent to

$$(Ri)_0 = k^2 (1-k^2), \quad (53)$$

so that $(Ri)_0$ has a maximum of $\frac{1}{4}$ for $k^2 = \frac{1}{2}$. For $(Ri)_0 > \frac{1}{4}$ all modes are stable, while for $(Ri)_0 = \frac{1}{4}$, instability occurs for $k^2 = \frac{1}{2}$. 
Howard (1961; see also Miles, 1961; Dutton and Fichtl, 1961) proved that if \( R_i \) exceeds a critical value (essentially \( \frac{1}{4} \)) throughout the fluid, then the motion must be stable in the inflectional sense, and that if the motion is unstable, then \( R_i \) must fall below the critical value somewhere in the fluid. Notice that this does not mean that instability must occur for \( R_i \) less than critical. Dutton (1976) discusses Howard's theorem in some detail. An analogous theorem exists for the instability of quasigeostrophic flow.
Lalas et al. (1976) considered the effect of a lower boundary on the discontinuous shear model. They found that the presence of the boundary increases the range of unstable wavelengths. They compared the phase speeds and periods of their fastest-growing modes with those of observed modes (see Table 3.1) finding fair agreement.

<table>
<thead>
<tr>
<th>Case 1</th>
<th>Case 2</th>
<th>Case 3</th>
<th>Case 4</th>
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<tr>
<td>(Ottersten et al., 1968)</td>
<td>(Hooke et al., 1973)</td>
<td>(Hooke et al., 1973)</td>
<td>(Hooke and Hardy, 1975)</td>
</tr>
</tbody>
</table>

| Bottom layer height \( h \) (km) | 11 | 0.12 | 2.5 | 9 |
| Velocity jump \( u_0 = \Delta \psi / \beta \) (m s\(^{-1}\)) | 10 | 1 | 7 | 28 |
| Brunt-Väisälä frequency \( \nu \) (s\(^{-1}\)) | \(2 \times 10^{-2}\) | \(2.7 \times 10^{-2}\) | \(2 \times 10^{-2}\) | \(1.3 \times 10^{-2}\) |
| Observed wavelength \( \lambda_o \) (km) | 6 | 0.330 | 2.7 | 15–20, 1.6 |
| Lindzen’s optimal \( \lambda_o \) (km) | 4.44 | 0.320 | 3.1 | 19.1 |
| Mode II, most unstable \( \lambda_o \) (km) | 4.46 | 0.320 | 3.05 | 20.0 |
| Mode III, most unstable \( \lambda_o \) (km) | 4.44 | 0.362 | — | — |
| Mode II, growth rate \( |u_o| \) (s\(^{-1}\)) | \(1.208 \times 10^{-3}\) | \(5.167 \times 10^{-3}\) | \(8.96 \times 10^{-4}\) | \(9.95 \times 10^{-4}\) |
| Mode III, growth rate \( |u_o| \) (s\(^{-1}\)) | \(1.241 \times 10^{-3}\) | \(1.45 \times 10^{-3}\) | — | — |
| Phase velocity\(^a\), observed (m s\(^{-1}\)) | not given | 2.9–3.8 | 20 | 47 |
| Phase velocity\(^b\), Mode II (m s\(^{-1}\)) | 23.6 | 3.67 | 23.5 | 53.2 |
| Phase velocity\(^b\), Mode III (m s\(^{-1}\)) | 25.9 | 3.98 | — | — |
| Period, observed\(^c\) (s) | not given | 100 | 152 | 300 |
| Period, Mode II (s) | 189 | 85 | 172 | 377 |
| Period, Mode III (s) | 172 | 91 | — | — |

\(^a\) This phase speed is relative to an observer stationary on the ground.

\(^b\) To calculate the phase speed of the wave, the mean wind velocity at the interface at \( z = h \) had to be used. From the data, the values extracted were 25, 3.5, 17 and 43 m s\(^{-1}\) for cases 1 to 4, respectively. There is considerable uncertainty as to the exact location of the interface and thus the previously mentioned mean wind values are only approximate.

\(^c\) The previously mentioned mean wind, with its uncertainty, had to be utilized here as well.
C. Inertial instability

The simplest analysis of inertial stability and instability proceeds from the consideration of a circular vortex. For such a vortex, the equation of radial motion takes the form

\[ \frac{d\nu_r}{dt} = - \frac{1}{\rho} \frac{\partial p}{\partial r} + r \omega^2, \]  

where \( r \) is the radius from the center of the vortex, and \( \omega \) is the angular velocity. The second term on the right-hand side of (1) represents the centripetal acceleration. According to (1), we can have an equilibrium in which the outward centripetal "force" is balanced by the radial component of the pressure gradient force. Defining the angular momentum by

\[ M = \rho \omega r^2, \]  

we can rewrite (1) as
\[
\frac{d\vec{r}}{dt} = -\frac{1}{2} \frac{\delta p}{\delta r} + \frac{M^2}{\rho^2 r^3}.
\] (3)

Suppose that we perturb the vortex, by radially displacing a ring of fluid, from \( r = r_A \) to \( r = r_B > r_A \). Since the parcel will conserve its angular momentum, we can write

\[
\left( \frac{d\vec{r}}{dt} \right)_B = -\frac{1}{2} \left( \frac{\delta p}{\delta r} \right)_B + \frac{MA}{\rho^2 r_B^3} \jmath.
\] (4)

for simplicity, we neglect any variations in \( \rho \). If the vortex was in equilibrium before we poked it, and if we assume that our perturbation does not disturb the pressure field, then we have

\[
-\frac{1}{2} \left( \frac{\delta p}{\delta r} \right)_B = -\frac{M_B}{\rho^2 r_B^3}.
\] (5)

Substitution of (5) into (4) gives

\[
\left( \frac{d\vec{r}}{dt} \right)_B = \frac{MA - MB}{\rho^2 r_B^3} \jmath.
\] (6)
According to (6), the displaced fluid ring will experience a restoring force if $M_B^2 > M_A^2$, i.e., if the square of the angular momentum increases outward. Otherwise, the ring will be accelerated further away from its original position. We conclude that a vortex in a homogeneous, inviscid fluid is unstable if

$$\frac{dM^2}{dt} < 0$$

(7)

More generally, we must consider the inertial stability of flows (not necessarily circular vortices) in stratified fluids. The stability criterion is remarkably similar to (7):

$$\frac{1}{r^2} \left( \frac{\partial M^2}{\partial r} \right) < 0$$

(8)

is the necessary condition for instability, where $r$ is the absolute radius of curvature of the flow, and the derivative is computed along an isentropic surface. Why an isentropic surface? Well, a particle perturbed in such a way that
it is forced off its original isentropic surface will experience a restoring force due to buoyancy, in a stably stratified fluid. So the optimal direction of perturbation, to achieve instability, is along an isentropic surface; the buoyancy force on the particle is then zero. In other words, if any perturbation is unstable, then a perturbation along an isentropic surface will be unstable.

It can be shown (e.g., Dutton, 1970) that the necessary condition for inertial instability can also be written as

\[ \xi_g + f < 0 \]

(9)

where \( \xi_g \) is the vertical component of the geostrophic vorticity, which will in many cases of interest be nearly equal to the true vorticity. According to (9), the flow can be inertially unstable when the absolute vorticity is negative.
Emanuel (1978) argued that inertial instabilities are identical to a type of Ekman layer instability whose interpretation had previously been obscure. He also suggested that inertial instability may help to organize some squall lines. Raymond (1978) reached similar conclusions. We discuss these theories further later in the course.
D. Ekman layers and their instabilities

1. Introduction

We define an Ekman layer to be any layer in which the primary balance in the momentum equation is between friction, the pressure gradient force, and the coriolis acceleration. Here we have not assumed any particular relationship between the frictional stress and the other variables of the problem. It is easy to show that in steady, non-adveective Ekman layers the mean flow always has a component towards low pressure.

Frictional flow  Geostrophic flow  Ekman flow
without rotation  (no friction)
2. Ekman pumping

The momentum equation

\[ \int_{z_s}^{z_B} \rho \left( \nabla \cdot (V - V_g) \right) dz = \frac{1}{f} \int_{z_s}^{z_B} \mathbf{k} \cdot \nabla \times (\Omega)_s \] (1)

for the horizontal component of motion in a steady non-advective Ekman layer can be integrated vertically to a level high enough so that \( \Omega \) vanishes, to give

\[ \int_{z_s}^{z_B} \rho \left( \nabla \cdot (V - V_g) \right) dz = -\frac{1}{f} \int_{z_s}^{z_B} \mathbf{k} \times (\Omega)_s \] (2)

or

\[ \int_{z_s}^{z_B} \rho \left( \nabla \cdot (V - V_g) \right) dz = -\frac{1}{f} \int_{z_s}^{z_B} \mathbf{k} \times (\Omega)_s \] (3)

This means that the vertically-integrated ageostrophic wind is determined by the surface stress. We can rewrite this as

\[ \int_{z_s}^{z_B} \rho \left( \nabla \cdot (V - V_g) \right) dz = -\frac{1}{f} \int_{z_s}^{z_B} \mathbf{k} \cdot \nabla \times (\Omega) s \] (4)
or, if \( z_B \) is horizontally uniform,

\[
\begin{align*}
\int_{z_s}^{z_B} \mathbf{\nabla} \cdot (\mathbf{\varphi} L) \, dz &= -\frac{1}{\tau} \nu \cdot \mathbf{\nabla} \times (\mathbf{\varphi} L) s. \quad (5)
\end{align*}
\]

Using continuity, and taking \( \omega s = 0 \), we obtain

\[
\left( \nu \mathbf{\omega} \right)_B = \frac{1}{\tau} \nu \cdot \mathbf{\nabla} \times (\mathbf{\varphi} L) s. \quad (6)
\]

The vertical mass flux at the top of a steady, non-advective Ekman layer is determined by the curl of the surface stress. This conclusion is quite independent of the nature of \( \mathbf{\varphi} \) — it can be viscous, in the molecular sense, or turbulent. Because the Ekman layer forces a convergence of mass and ascent, or a divergence of mass and descent, depending on the sign of \( \frac{1}{\nu} \nu \cdot \mathbf{\nabla} \times (\mathbf{\varphi} L) s \), people speak of "Ekman pumping."
Emanuel (1978) argued that inertial instabilities are identical to a type of Ekman layer instability whose interpretation had previously been obscure. He also suggested that inertial instability may help to organize some squall lines. Raymond (1978) reached similar conclusions. We discuss these theories further later in the course.
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\[ -F \quad -f_k x v \quad -f_k x v \quad -F \]

Frictional flow \quad Geostrophic flow \quad Ekman flow without rotation (no friction)
2. Ekman pumping

The momentum equation

\[ f \kappa \times \rho (\mathbf{V} - V_0) = \frac{\partial \vec{z}}{\partial z} \]  

(1)

for the horizontal component of motion in a steady, non-adveective Ekman layer can be integrated vertically to a level high enough so that \( \vec{z} \) vanishes, to give

\[ f \kappa \times \int_{z_s}^{z_B} \rho (\mathbf{V} - V_0) \, dz = -\left( \tau \right)_s \]  

(2)

or

\[ \int_{z_s}^{z_B} \rho (\mathbf{V} - V_0) \, dz = -\frac{1}{f} k \times (\tau)_s \]  

(3)

This means that the vertically-integrated ageostrophic wind is determined by the surface stress. We can rewrite this as

\[ \nabla \cdot \int_{z_s}^{z_B} \rho (\mathbf{V} - V_0) \, dz = -\frac{1}{f} k \cdot \nabla (\tau)_s \]  

(4)
or, if $z_{B}$ is horizontally uniform,

\[
\int_{z_{B}}^{z_{S}} \mathbf{v} \cdot (\mathbf{r} \mathbf{v}) \, dz = - \frac{1}{f} \mathbf{k} \cdot \nabla x (z) \mathbf{S}. \tag{5}
\]

Using continuity, and taking $\omega_{S} = 0$, we obtain

\[
(\mathbf{g} \omega)_{B} = \frac{1}{f} \mathbf{k} \cdot \nabla x (z) \mathbf{S}. \tag{6}
\]

The vertical mass flux at the top of a steady, nonadvective Ekman layer is determined by the curl of the surface stress. This conclusion is quite independent of the nature of $\frac{z_{B}}{f}$ — it can be viscous, in the molecular sense, or turbulent. Because the Ekman layer forces a convergence of mass and ascent, or a divergence of mass and descent, depending on the sign of $\frac{1}{f} \mathbf{k} \cdot \nabla x (z) \mathbf{S}$, people speak of "Ekman pumping."
3. Laminar Ekman layers and the "Ekman spiral"

In laminar flows, which can be produced in the laboratory, the frictional stress is proportional to the rate of strain:

\[ \tau = \mu \frac{\partial V}{\partial z}, \]  

(7)

where \( \mu \) is a material property, which we assume to be constant. Then the steady, non-adveective equation of motion becomes

\[-f k x (V - V_g) + \nu \frac{\partial^2 V}{\partial z^2} = 0,\]  

(8)

where \( \nu = \mu / \rho \), and we neglect the slight variation of \( \nu \) with \( z \). Let

\[ \hat{V} \equiv (u - u_g) + i (\omega - u_g) \]  

(9)

and assume that \( V_g \) is uniform with height (i.e., that the "thermal wind" vanishes).
Then (8) becomes

$$(-2i + D^2 \frac{\partial^2}{\partial z^2}) \hat{V} = 0$$

(10)

where

$$D^2 = \frac{2\gamma}{f}$$

(11)

Notice that $D$ is a length, which is roughly the "depth" of the Ekman layer. Since $(1+i)^2 = 2i$, the solution of (10) is

$$\hat{V} = A \exp[(1+i)z/D] + B \exp[-(1+i)z/D]$$

(12)

where $A$ and $B$ are complex. Requiring that

$$\hat{V} = -ug - i\omega g \text{ for } z = 0$$

(13)

and

$$\hat{V} \to 0 \text{ for } z \to \infty$$

(14)
we find that

\[ A = 0 \]  \hspace{1cm} (15) \]

and

\[ B = -u_g - v_g \]  \hspace{1cm} (16) \]

Back substitution then shows that

\[ u = u_g - e^{-z/D} \left[ u_g \cos \left( \frac{z}{D} \right) + v_g \sin \left( \frac{z}{D} \right) \right] \]

(17)

and

\[ \sigma = u_g - e^{-z/D} \left[ u_g \cos \left( \frac{z}{D} \right) - v_g \sin \left( \frac{z}{D} \right) \right] \]

(18)

We conclude that the velocity vector "spirals" with height.

The theory developed above is in very poor agreement with atmospheric observations. The predicted wind veering is excessive, the predicted depth of the friction layer bears little relation to observed depths, etc. By modifying the lower boundary condition, allowing \( u \) and \( v \) to vary with height, and so on, it is possible to obtain improved agreement, but the theory is fundamentally flawed in that it can't explain observed countergradient momentum transport in the atmosphere.
4. Ekman layer instabilities

Lilly (1960) considered the stability of an incompressible, viscous rotating flow, governed by

\[
\frac{d \vec{V}}{dt} = -\nabla (\frac{F}{\rho}) - 2\Omega \hat{k} \times \vec{V} + \nu \nabla^2 \vec{V}
\]

\[
\nabla \cdot \vec{V} = 0
\]

\[
\vec{V} = 0 \text{ at } z = 0, \quad \nabla \cdot \vec{V} = 0 \text{ at } z = \infty.
\]

He assumed that the mean flow was steady:

\[
\vec{V} \cdot \nabla \vec{V} + \nabla \left( \frac{F}{\rho} \right) + 2\Omega \hat{k} \times \vec{V} = \nu \nabla^2 \vec{V}
\]

\[
\nabla \cdot \vec{V} = 0
\]

and studied the growth of small disturbances:

\[
\frac{d \vec{V}^*}{dt} + \vec{V}^* \cdot \nabla \vec{V} + \vec{V} \cdot \nabla \vec{V}^* + \nabla \left( \frac{F}{\rho} \right) + 2\Omega \hat{k} \times \vec{V}^* = \nu \nabla^2 \vec{V}^*
\]
For small $R_0$, the advection terms in the mean flow equations can be dropped. Then

\[ v = i\omega g \left( 1 - e^{-z/D} \cos \frac{z}{D} \right) + j\omega g e^{-z/D} \sin \frac{z}{D} \]

(22)

where

\[ \omega g = \frac{1}{2\Omega \rho} \frac{d\Omega}{dr} \]

(23)

and

\[ D = \left( \frac{\varphi}{\Omega} \right)^{1/2} \]

(24)

Notice that $V_g$ has been taken to lie along the $x$-axis. We write the perturbation equations in Cartesian coordinates, neglecting curvature ($\text{radius } \gg D$). Let there be no variation along the $x'$-axis, where $x'$ is at an angle $\varepsilon$ to the geostrophic flow:

\[ \frac{d}{dx'} = 0 \]
Then define new, nondimensional variables as follows:

\[
\begin{align*}
    x' &= \frac{(x \cos \theta + y \sin \theta)}{D} \\
    y' &= \frac{(-x \sin \theta + y \cos \theta)}{D} \\
    z' &= \frac{z}{D} \\
    u' &= \frac{(u \cos \theta + v \sin \theta)}{U} \\
    v' &= \frac{(-u \sin \theta + v \cos \theta)}{U} \\
    w' &= \frac{w}{U} \\
    \pi' &= \frac{p^*}{\rho U^2} \\
    t' &= \frac{t}{U} \frac{1}{D}
\end{align*}
\]

Our equations become

\[
Re \left( \frac{\partial u'}{\partial t'} + V \frac{\partial u'}{\partial y'} + w' \frac{\partial u'}{\partial z'} \right) - 2u' = \frac{\partial^2 u'}{\partial y'^2} + \frac{\partial^2 u'}{\partial z'^2}
\]

(26)

\[
Re \left( \frac{\partial u'}{\partial t'} + V \frac{\partial u'}{\partial y'} + w' \frac{\partial V}{\partial z'} + \frac{\partial \pi'}{\partial y'} \right) + 2u' = \frac{\partial^2 u'}{\partial y'^2} + \frac{\partial^2 u'}{\partial z'^2}
\]

(27)

\[
Re \left( \frac{\partial w'}{\partial t'} + V \frac{\partial w'}{\partial y'} + \frac{\partial \pi'}{\partial z'} \right) = \frac{\partial^2 w'}{\partial y'^2} + \frac{\partial^2 w'}{\partial z'^2}
\]

(28)

\[
\frac{\partial u'}{\partial y'} + \frac{\partial w'}{\partial z'} = 0
\]

(29)

where

\[
Re = \frac{\rho U D}{\nu_L} = \frac{U}{(\nu L)^{1/2}}
\]

(30)
\[ \frac{d}{dz'} (u' \cos \varepsilon + v' \sin \varepsilon) / \sigma_g = \cos \varepsilon - e^{z'} \cos (z' + \varepsilon) \quad (31) \]

\[ \sqrt{\left( u' \sin \varepsilon + v' \cos \varepsilon \right)} / \sigma_g = -\sin \varepsilon + e^{z'} \sin (z' + \varepsilon) \quad (32) \]

The b.c.'s are

\[ u' = v' = \omega' = 0 \quad \text{at} \quad z' = 0 \]

\[ \frac{du'}{dz'} = \frac{dv'}{dz'} = \omega' = 0 \quad \text{at} \quad z' = \infty \quad (33) \]

Define \( \phi' \) by

\[ \phi' = -\frac{d\phi'}{dz'}, \quad \omega' = \frac{d\phi'}{dy} \quad (34) \]

Then

\[ \phi' = \frac{dw'}{dy} = \frac{dx'}{dy} = \frac{d^2x'}{dy^2} + \frac{d^2x'}{dz'^2} \quad (35) \]

obeys

\[ R(\frac{d\phi'}{dz'} + V \frac{d\phi'}{dy} - \omega' \frac{d^2V}{dz'^2}) - 2 \frac{dw'}{dz'} = \frac{d^2\phi'}{dy^2} + \frac{d^2\phi'}{dz'^2} \quad (36) \]

Assume that

\[ \phi' = \phi(z') e^{i\alpha (y' - ct')} \quad (37) \]

\[ u' = \mu(z') e^{i\alpha (y' - ct')} \]
Our equations become

\[ \varphi''' - 2z^2 \varphi'' + \alpha^4 \varphi - \text{i} \alpha R \left[ (V - c)(\varphi'' - \alpha^2 \varphi) - V \varphi \right] + 2\mu' = 0 \]

(38)

\[ \mu - \alpha^2 \mu - \text{i} \alpha R \left[ (V - c)\mu + U \varphi \right] - 2\varphi' = 0 \]

(39)

\[ \varphi = \varphi' = \mu = 0 \text{ at } z' = 0 \]

(40)

\[ \varphi = \varphi'' = \mu' = 0 \text{ at } z' = \infty \]

(41)

Here \( d/dz' \) denotes differentiation with respect to \( z' \).

This is an eigenvalue problem. For given \( \alpha \), \( x \), and \( \varepsilon \), we seek eigenfunctions \( \varphi \) and \( \mu \) with eigenvalues \( c_r \) and \( c_i \). The growth rate is \( c_i \). The solutions are obtained numerically.

Lilly found two types of instability, which he called parallel and inflectional, respectively. The first sets in at a critical Reynolds number of 6.5, and is characterized by a negative orientation angle. Emanuel (1978) has shown that the parallel instability is of the inertial type. The inflectional instability sets in at a critical Reynolds number of 110, and has a positive orientation angle. Both types are observed in the laboratory (e.g., Greenspan, 1968).
PARALLEL INSTABILITY

Re_c = 6.5

axis of symmetry

0 to 8°

INFLECTIONAL INSTABILITY

Re_c = 110

axis of symmetry

14.6°
For sufficiently high Reynolds numbers, the inflectional instability grows much faster than the parallel instability, so that the latter is probably not very important in the PBL where the (effective) Reynolds number is believed to be on the order of $10^5$ or more.

The existence of Ekman layer instabilities is sometimes cited (e.g., Kraus, 1972) as a partial explanation for the absence of Ekman spirals in the atmosphere.
Many theoretical and laboratory studies have been done on the problem of thermal convection in a layer of viscous, conducting fluid heated from below. While the problem is of interest in itself it also has application to the atmosphere. The cellular convection patterns observed in the laboratory strongly resemble mesoscale convection patterns as seen in satellite photos. Another application is the study of the transition from laminar to turbulent flow - this takes place in a series of discrete transitions as the heating rate from below is increased, proceeding from rest to steady cells to oscillating cells and finally to fully turbulent flow.

We use the Boussinesq approximation, and let the mean wind and mean potential temperature obey

\[ \bar{u} = \Lambda z \quad \text{(1)} \]

and

\[ \Theta_0 = \Theta_{oo} - \Gamma' z \quad \text{(2)} \]

where \( \Lambda, \Theta_{oo}, \text{ and } \Gamma' \) are constants. Rotation is included, but we assume for simplicity that the axis of rotation is vertical. (Chaudrasekhar, 1961, considers the more general case, briefly.) Linearizing for small perturbations, we have
\[
\left( \frac{\partial}{\partial t} + \Lambda z \frac{\partial}{\partial x} - u \nabla^2 \right) u' - 2\Omega w' = -\frac{\partial}{\partial x}\left( \frac{\partial u'}{\partial z} \right), \\
\left( \frac{\partial}{\partial t} + \Lambda z \frac{\partial}{\partial x} - u \nabla^2 \right) w' + 2\Omega w' = -\frac{\partial}{\partial z}\left( \frac{\partial w'}{\partial x} \right), \\
\left( \frac{\partial}{\partial t} + \Lambda z \frac{\partial}{\partial x} - u \nabla^2 \right) w' = -\frac{\partial}{\partial y}\left( \frac{\partial u'}{\partial z} \right), \\
\frac{\partial u'}{\partial x} + \frac{\partial w'}{\partial y} + \frac{\partial u'}{\partial z} = 0, \\
\left( \frac{\partial}{\partial t} + \Lambda z \frac{\partial}{\partial x} - u \nabla^2 \right) T' = -w' \Gamma. 
\]

Here \( \nabla^2 \) is the three-dimensional Laplacian, \( \Omega \) is the angular velocity of rotation, and \( \kappa = \frac{1}{\text{To}} \). Elimination of \( u' \), \( w' \), and \( \rho' \) among (3–6) gives

\[
\left( \frac{\partial}{\partial t} + \Lambda z \frac{\partial}{\partial x} - u \nabla^2 \right) \nabla^2 w' = g x \nabla_h^2 T' - 2\Omega \frac{\partial \zeta'}{\partial z},
\]

where \( \nabla_h^2 \) is the two-dimensional Laplacian, and \( \zeta' = \frac{\partial u'}{\partial x} - \frac{\partial w'}{\partial y} \) is the perturbation vorticity, which obeys
\[
\left( \frac{\partial}{\partial t} + \Lambda z \frac{\partial}{\partial x} - v \frac{\partial^2}{\partial z^2} \right) \xi' = \Lambda \frac{\partial \xi'}{\partial y} + 2\Omega \frac{\partial \xi'}{\partial z}.
\]

(9)

The \(\xi' / dy\) term of (9) represents the effects of twisting, while the \(\xi' / dz\) term represents the stretching of vortex tubes. For convenience, we note the following relations:

\[
\nabla_H^2 u' = - \frac{\partial \xi'}{\partial y} - \frac{\partial \xi'}{\partial x} \frac{\partial}{\partial z}
\]

(10)

\[
\nabla_H^2 v' = \frac{\partial \xi'}{\partial x} - \frac{\partial \xi'}{\partial y} \frac{\partial}{\partial z}
\]

(11)

If (7), (8), and (9) are solved simultaneously for \(T', u',\) and \(\xi',\) then (10) and (11) can be used to obtain the horizontal velocity field.

We postpone discussion of boundary conditions.
Assume solutions of the form

\[
\begin{bmatrix}
\omega' \\
T' \\
\delta' \\
\mu' \\
\sigma'
\end{bmatrix} = \begin{bmatrix}
W(2) \\
\Theta(2) \\
Z(2) \\
U(2) \\
V(2)
\end{bmatrix} \exp\left[i(kx+ly)+\sigma t\right]
\]

(12)

Let the depth of the motion be \( h \). We nondimensionalize as follows:

\[
\begin{align*}
\sigma &= \sigma^*/h^2 \\
z &= z^*/h \\
k_z &= k_z^*/h \\
l &= l^*/h \\
U &= U^*/h \\
V &= V^*/h \\
W &= W^*/h \\
Z &= Z^*/h^2 \\
\Theta &= \Theta^*/\Gamma h
\end{align*}
\]

(13)
Define the following nondimensional combinations, for later convenience:

- Prandtl number \( Pr = \frac{\nu}{\kappa} \)
- Rayleigh number \( Ra = \frac{g \alpha \Gamma h^4}{(\kappa \nu)} \)
- Reynolds number \( \text{Re} = \frac{\Delta h^2}{\nu} \)
- Richardson number \( \text{Ri} = \frac{g \alpha \Gamma}{\Delta^2} \) 
  \[ = \frac{Ra}{(Pr \cdot \text{Re}^2)} \]
- Ekman number \( E = \frac{\nu}{(2 \pi^2 h^2)} \)
- Taylor number \( T = \frac{4 \pi^2 h^4}{\omega^2} \) 
  \[ = E^{-2} \]

From this point, we treat \( x \) as a constant.

We now use (12-14) in (7-9), and suppress the \( \ast \) notation, for convenience. We also invoke the principle of the exchange of stabilities, seeking the marginal state in which \( r = 0 \).

This procedure yields
\[ \left[ i \text{Re} k \frac{z}{Pr} \left( \frac{d^2}{dz^2} - k^2 - q^2 \right) \right] \Theta - W = 0, \tag{15} \]

\[ \left[ i \text{Re} k \frac{z}{Pr} \left( \frac{d^2}{dz^2} - k^2 - q^2 \right) \right] \left( \frac{d^2}{dz^2} - k^2 - q^2 \right) W = - \frac{Ra}{Pr} \left( k^2 + q^2 \right) \Theta + E^{-1} \frac{dZ}{dz}, \tag{16} \]

and

\[ \left[ i \text{Re} k \frac{z}{Pr} \left( \frac{d^2}{dz^2} - k^2 - q^2 \right) \right] Z = \frac{dW}{dz}, \tag{17} \]

\[ i \lambda \text{Re} W - \frac{E^{-1}}{dW} = \frac{dW}{dz}. \tag{17} \]

To proceed further, we consider some special cases.
1. Free convection, without rotation

We set $Re = 0$ and $Ec = 0$, and eliminate $\Theta$ between (15) and (16), to obtain a single sixth-order ordinary differential equation for $W$:

$$\left( \frac{d^2}{dx^2} - k^2 - \lambda^2 \right) W = -Ra (k^2 + \lambda^2) W$$

(18)

This is an eigenvalue problem. The only parameter controlling the motion is $Ra$. We now give a heuristic interpretation of $Ra$.

Suppose that $\Gamma = 0$, so that $Ra = 0$.

Then we expect no motion. For very small $Ra$, the thermal conductivity alone is enough to transport sufficient heat to maintain a steady state. Any motions would be very weakly driven—so weakly that viscosity would overwhelm and suppress them. The rate of kinetic energy generation, per unit mass, is

$$\frac{g x \lambda \theta^2}{k/n^2} \sim g x u_s. \frac{\Gamma u_s}{k/n^2} = \frac{g x \lambda u_s^2 h^2}{k}$$

(19)
where we recognize that the thermal conductivity tends to reduce \( \Theta \). The rate of kinetic energy dissipation, per unit mass, is

\[
(\nabla \cdot \mathbf{w}) \cdot \mathbf{w} \sim \frac{\nu w^2}{h^2}
\]

So the ratio of the generation rate to the destruction rate is measured by

\[
\frac{g\lambda \nu w^2 h^2 / k}{\nu h^2 / h^2} = \frac{g\lambda \nu h^4}{k\nu} = Ra
\]

This rough argument is given as an aid to intuition.

For sufficiently large \( Ra \), it becomes possible to generate kinetic energy rapidly enough to overcome dissipation. Motions then ensue. We now determine the scale of the motion, and the critical value of \( Ra \), but we must first choose our boundary conditions.
At a rigid boundary, we have no slip:

\[ U = V = W = 0 \]  \hspace{1cm} (22a)

From continuity, it then follows that

\[ \frac{dW}{dz} = 0 \]  \hspace{1cm} (22b)

Also, we have

\[ \varepsilon_z = 0 \]  \hspace{1cm} (22c)

And

\[ \Theta = 0 \]  \hspace{1cm} (22d)

At a free surface, we have no stress. Then

\[ \frac{dU}{dz} = \frac{dV}{dz} = 0 \]  \hspace{1cm} (23a)

Again, we have
\[ W = 0 \]  \hspace{2cm} (23b)

Continuity with (25a) gives \( \frac{d^2 W}{dz^2} = 0 \)  \hspace{2cm} (23c)

Also,
\[ \frac{d Z_i}{dz} = 0 \]  \hspace{2cm} (23d)

and, finally, we again take
\[ \Theta = 0 . \]  \hspace{2cm} (23e)

In the case of the free surface, we apply our boundary conditions at the unperturbed position of the surface, as an approximation. (The correction term is of second order.) Notice that the vorticity equation, (17), reduces simply to
\[ \left( \frac{d^2}{dz^2} - k^2 - l^2 \right) Z_i = 0 . \]  \hspace{2cm} (24)
With any combination of the fixed and free boundary conditions listed, we find that

\[ Z_1 = 0 \]  \hspace{1cm} (25)

i.e., the motions involve no vertical vorticity. Either shear or rotation will modify this result.

The exact solutions for three possible combinations of boundary conditions are given below. For each case, the solution can be substituted back into (18), to yield the total wave number as a function of \( Ra \), on the boundary between growing and damped disturbances. For a given wave number, we have instability for \( Ra > (Ra)_{\text{neutral}} \). As \( Ra \) is increased from zero, the first modes to appear will be those for which \( (Ra)_{\text{neutral}} \) is minimized. This occurs for \( Ra = (Ra)_{\text{crit}} \).

We seek solutions of the general form

\[ W = Ae^{\delta z} + Be^{-\delta z} \]  \hspace{1cm} (26)

where

\[ (\delta^2 - a^2) = -Ra \cdot a^2 \]  \hspace{1cm} (27)

and

\[ a^2 = k^2 + \ell^2 \]  \hspace{1cm} (28)
for convenience. After solving (29) for $g$ as a function of $a^2$ and $Ra$, we impose the boundary conditions to determine the relationship between $a^2$ and $Ra$. Details are given by Chandrasekhar (1961).

We now consider various combinations of boundary conditions, case by case.

a) Both boundaries "fixed," at $z = \pm \frac{1}{2}$

The iterative solution of a transcendental equation yields $Ra$ as a function of $a^2$ and it is found that

$$ (Ra)_{\text{crit}} = 17.08 \quad (29a) $$

and that the first mode to appear has

$$ a = (a)_{\text{crit}} = 3.117 \quad (29b) $$

b) One fixed and one free boundary.

The solution is again obtained numerically. The result is

$$ (Ra)_{\text{crit}} = 1100.65 \quad (30a) $$

$$ (a)_{\text{crit}} = 3.6825 \quad (30b) $$
c) Two free boundaries

In this case, (26) can be simplified to

\[ W = A \sin \frac{n \pi z}{l}; \quad n = 1, 2, \ldots \quad (31) \]

which leads to

\[ Ra = \frac{(n^2 \pi^2 + a^2)^{3/2}}{a^2} \quad (32) \]

For a given \( a \), \( Ra \) is minimized for \( n = 1 \).

We find that

\[ (Ra)_{\text{crit}} = 657.5 \quad (33a) \]

and

\[ (a)_{\text{crit}} = \frac{\pi}{2} \approx 4.94 \quad (33b) \]

Notice that, as the boundaries are "fixed," \( (Ra)_{\text{crit}} \) decreases, and \( (a)_{\text{crit}} \) increases.
2. What determines the platforms?

Our analysis up to this point has given us the vertical structure of the motions at the onset of instability, but it has told us nothing about the platform of the motions, except that it is a superposition of normal modes of the form (12). Suppose that the convection covers an infinite plane, and that the boundary conditions and initial conditions are completely uniform over this plane. Then we expect the entire plane to be covered by convection, with no "gaps." Furthermore, since there are no preferred directions, the boundaries of the cells (say, the boundaries on which \( u = \text{maximal} \)) should be surfaces of symmetry. So we expect the plane to be covered by regular polygons. Let \( n \) be the number of sides, so that the angle between adjacent sides is \( \frac{\pi (n-2)}{n} \). This angle must divide \( 2\pi \) \( m \) times, where \( m \) is an integer:

\[
\frac{2\pi}{m} = \pi \left( 1 - \frac{2}{n} \right) \quad (34)
\]

Since \( n \) and \( m \) must be integers, the only possibilities are \( n = 3, 4, 6 \) (\( m = 4, 4, 3 \)).
So the polygons must be triangles, rectangles (including bands as infinitely wide rectangles), or hexagons. The mathematical descriptions of the circulations in such cells have been worked out, and are summarized by Chandrasekhar (1961).

Triangular cells exist in any field of hexagonal cells, so the two types are not really distinct.

Chandrasekhar (1961)
A particle inside OMN can't get out; there are no transverse velocities at the walls of this triangle. This is also true of triangle OCD (and its neighbors). But the vertical motions at C and D are in the opposite sense of that at O, so the "cell" OCD is not symmetric. In contrast, the vertical motions at M, N, and O are all identical; this cell is symmetric. So only MNO is a cell in the sense we wish to consider here.

Since triangular and hexagonal cells are equivalent, it suffices to consider only hexagonal and rectangles separately. Both have been produced in the laboratory. What determines which will occur in a given experiment? Schlüter, Lortz, and Busse (1965) showed that, near \( \text{Re}_{\text{crit}} \), finite-amplitude hexagonal cells are unstable with respect to bands, whereas finite-amplitude bands are stable. Hence only bands should be observed.
But in the laboratory, hexagons are actually produced more frequently than rolls. To understand this, we note, following Krishnamurti (1970), that the direction of the roll circulation cannot be unambiguously defined, since the apparent direction can be reversed simply by a translation of the observer.

A FIELD OF ROLLS

In contrast, motion in hexagonal cells can be either "closed" (up in the center) or "open" (down in the center), and the two possibilities are physically distinguishable.
In the closed cell, the region of rising motion is simply connected (for a particular cell), while the region of sinking motion has holes in it (one "hole" for each cell). The reverse is true for open cells. These are objective distinctions.

So for the rolls, "up" and "down" can't be distinguished by looking at the stream lines. But for either open or closed cells, up and down are distinguishable. This vertical asymmetry must be associated with a corresponding asymmetry in the boundary conditions. On the other hand, if the boundary conditions are perfectly symmetric, then we expect a symmetric circulation.

In short, we should expect hexagons to be preferred (and to be stable) when the boundary conditions are asymmetric. And since this can easily happen by accident (asymmetries are easier to arrange than symmetries), we are no longer surprised that hexagons appear more often, even near ($Ra$)$_{cr}$.

There are lots of ways to introduce asymmetries. One is to impress a mean vertical motion field. Another is to let the viscosity and/or thermal conductivity vary with temperature. A third is to change the
boundary temperatures with time, so that the basic state temperature profile is asymmetric, since it takes the interior of the fluid a finite time to "find out" that the boundary temperature has changed. More than one asymmetry may be present simultaneously, perhaps with competing effects.

**BOUNDARY T's FIXED**

**BOTH BOUNDARY T’s INCREASING WITH t**

Let's suppose, for the sake of discussion, that the rolls and hexagons seen in cloud pictures or on radar (etc.) are the same kind of animals as the laboratory circulations that we've been studying. It's hard to believe that the "boundary conditions" provided by the synoptic flow are symmetric very often. Yet rolls are much more common than hexagons.

To understand the probable explanation of this discrepancy, we must consider the effects of shear. This we postpone, briefly.
3. The effects of notation

Now let $E \neq 0$; but continue to consider $Re = 0$. If the principle of the exchange of stabilities remains valid, then corresponding to (18) we will have

$$\left[ \left( \frac{d^2}{dz^2} - k^2 - \lambda^2 \right)^3 + T \frac{d^2}{dz^2} \right] W = - Ra (k^2 + \lambda^2) W$$

(35)

The Prandtl number again drops out, but now both the Taylor number and the Rayleigh number influence the motion. Two boundary conditions are

$$W = 0 \quad \text{if} \quad z = 0, 1 \quad \text{(36)}$$

$$\Theta = 0$$

In view of (16) (with $Re = 0$) we can rewrite these as

$$W = 0$$

$$\left( \frac{d^2}{dz^2} - k^2 - \lambda^2 \right)^2 W - E \int \frac{dZ}{d^2} = 0 \quad \text{if} \quad z = 0, 1 \quad \text{(37)}$$
From (37), we see that, as a consequence of the rotation, we can't obtain a solution for \( W \) without considering the vorticity equation (17). This means that, whereas the order of our system was six without rotation \( \text{[see (18)]} \), it is eight with rotation. The remaining boundary conditions are either

\[
Z = 0 \quad \text{and} \quad \frac{dW}{dz} = 0 \quad (\text{rigid surface})
\]

or

\[
\frac{dZ}{dz} = 0 \quad \text{and} \quad \frac{d^2W}{dz^2} = 0 \quad (\text{free surface})
\]

By solving the eigenvalue problem, we can obtain the critical Rayleigh number for a given Taylor number, and also the scale of the first motions to appear. The details of this analysis are omitted here, but are given by Chaudhri Sekhar (1961). The results are summarized in the tables below.
Critical Rayleigh numbers and wave numbers of the unstable modes at marginal stability for the onset of stationary convection when both bounding surfaces are free

<table>
<thead>
<tr>
<th>$T$</th>
<th>$a_0$</th>
<th>$R_e$</th>
<th>$T$</th>
<th>$a_0$</th>
<th>$R_e$</th>
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<td>0</td>
<td>2.233</td>
<td>6.576 $\times 10^3$</td>
<td>3 $\times 10^4$</td>
<td>10.45</td>
<td>4.257 $\times 10^4$</td>
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<tr>
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<tr>
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<td>4.947 $\times 10^4$</td>
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<td>4.037 $\times 10^4$</td>
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Critical Rayleigh numbers and related constants for the case when one bounding surface is rigid and the other is free and the onset of instability is as stationary convection

<table>
<thead>
<tr>
<th>$T$</th>
<th>$a_0$</th>
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<td>+0.06235</td>
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<td>+0.06378</td>
<td>+0.05539</td>
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<tr>
<td>1.875 $\times 10^3$</td>
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<td>1.695 $\times 10^3$</td>
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<td>+0.05539</td>
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<td>1.6387 $\times 10^3$</td>
<td>+0.07053</td>
<td>+0.05539</td>
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<td>7.230 $\times 10^3$</td>
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<td>+0.05539</td>
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<td>+0.05539</td>
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<td>+0.05539</td>
</tr>
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<td>11.05</td>
<td>6.177 $\times 10^4$</td>
<td>6.0874 $\times 10^4$</td>
<td>+0.07053</td>
<td>+0.05539</td>
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<tr>
<td>1.875 $\times 10^3$</td>
<td>12.00</td>
<td>8.218 $\times 10^4$</td>
<td>8.2352 $\times 10^4$</td>
<td>+0.07053</td>
<td>+0.05539</td>
</tr>
<tr>
<td>10^3</td>
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<td>+0.05539</td>
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<td>10^3</td>
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<td>3.7619 $\times 10^5$</td>
<td>+0.07053</td>
<td>+0.05539</td>
</tr>
</tbody>
</table>

Critical Rayleigh numbers and related constants for the case when both bounding surfaces are rigid and the onset of instability is as stationary convection

<table>
<thead>
<tr>
<th>$T$</th>
<th>$a_0$</th>
<th>$R_e$</th>
<th>$T$</th>
<th>$a_0$</th>
<th>$R_e$</th>
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<td>10</td>
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<td>1.7710 $\times 10^3$</td>
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<td>1.764 $\times 10^3$</td>
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<td>+0.02884</td>
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<tr>
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<td>1.9405 $\times 10^3$</td>
<td>2.3171 $\times 10^3$</td>
<td>+0.02884</td>
</tr>
<tr>
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<td>2.1357 $\times 10^3$</td>
<td>2.5389 $\times 10^3$</td>
<td>+0.02884</td>
</tr>
<tr>
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<td>2.3389 $\times 10^3$</td>
<td>3.4766 $\times 10^3$</td>
<td>+0.02255</td>
</tr>
<tr>
<td>5,000</td>
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<td>3.5692 $\times 10^3$</td>
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<td>4.7131 $\times 10^3$</td>
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<td>+0.02255</td>
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<tr>
<td>30,000</td>
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<td>8.356 $\times 10^3$</td>
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<td>7.20</td>
<td>1.674 $\times 10^4$</td>
<td>1.6741 $\times 10^4$</td>
<td>1.6741 $\times 10^4$</td>
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<tr>
<td>10^4</td>
<td>10.80</td>
<td>7.153 $\times 10^4$</td>
<td>7.1136 $\times 10^4$</td>
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<tr>
<td>10^4</td>
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<td>3.4636 $\times 10^6$</td>
<td>3.4636 $\times 10^6$</td>
<td>+0.02255</td>
</tr>
</tbody>
</table>

Chandrasekhar (1941)
In each case, the critical Rayleigh number is increased by rotation, i.e., the rotation has a stabilizing influence. The wave number of the marginal disturbance increases with Taylor number; in other words, the scale decreases as the result of rotation. For large Taylor number, we have

$$\text{Ra}_c \sim T^{2/3}$$
$$a_c \sim T^{1/6} \quad T \to \infty$$

(40)

The stabilizing effect of rotation can be interpreted in terms of the Taylor-Proudman theorem. The vector vorticity $\mathbf{\omega}$ of an inviscid Boussinesq flow is governed by

$$\frac{\partial \mathbf{\omega}}{\partial t} + \mathbf{v} \times \left( \mathbf{v} \times \left( \mathbf{\omega} + 2 \mathbf{\Omega} \right) \right) = 0$$

(41)

For a steady flow which is slow enough that the nonlinear terms of (41) are negligible, we have
\( \nabla \times (\nabla \times \mathbf{\Omega}) = 0. \)  \hspace{4cm} \text{(42)}

Using the facts that \( \mathbf{\Omega} \) is a constant vector and that \( \nabla \cdot \mathbf{\Omega} = 0 \), we find from (42) that

\( (\mathbf{\Omega} \cdot \nabla) \mathbf{\Omega} = 0, \)  \hspace{4cm} \text{(43)}

i.e. the velocity is uniform in the direction of the rotation vector. This is the Taylor-Proudman theorem. When \( \mathbf{\Omega} \) has a vertical component, convective motions are inconsistent with (43), since there has to be inflow at some levels and outflow at others, i.e., the velocity cannot be uniform with height. Thus, convective motions tend to be inhibited by rotation.

Chaudhasekhar (1961) shows that for \( Pr \leq 1 \) and for sufficiently large Taylor numbers the principle of the exchange of stabilities is invalid; the convective motions set in as overstable oscillations. For \( Pr > 1 \), overstability cannot occur.
Now suppose again that the cloud patterns visible in satellite photographs are associated with mesoscale convective cells whose dynamics is essentially described by the theory we've been discussing here, provided that $K$ and $\nu$ are interpreted as "eddy viscosities." Are these mesoscale motions influenced appreciably by the earth's rotation? We can answer this question by estimating the appropriate Taylor number, and referring to our tables. Taking $\nu \approx 10^{-7} \text{ m}^2 \text{s}^{-1}$, $2\Omega \approx 10^{-4} \text{ s}^{-1}$, and $h \approx 2 \times 10^3 \text{ m}$, we find that

\[ T \approx \frac{4\Omega^2 h^4}{\nu^2} \approx 10 \]  

(44)

Inspection of our tables shows that rotation becomes important only for $T \geq 10^2$. Mesoscale cells would not be strongly influenced by rotation.
4. The effects of shear

Now consider (15-17), with Re ≠ 0 but E' = 0. Since Z drops out of (16), we can solve (15) and (16) simultaneously for θ and W. But because Re ≠ 0, Pr will not drop out of the problem; the motion is influenced by the three parameters Re, Pr, and Ra. According to (14), these three parameters can be combined to obtain the Richardson number, Ri. We choose to discuss the problem in terms of Ri, Ra, and Pr, and we shall consider only Pr = 7, which corresponds to water. The discussion is based on the paper of Asai (1970).

Notice that, although k and l appear symmetrically in (15-17) for Re = 0, they appear asymmetrically for Re ≠ 0. This means that the ratio k/l will influence the growth rate, and other aspects of the solution, when shear is present. We anticipate that, for given Ra, Ri, and Pr, the most rapid growth will occur for a particular value of k/l.
Asai (1970) solved (15) and (16) numerically, with the terms restored to obtain growth rates for a variety of conditions, as discussed below. We have already mentioned that Pr = 7 was assumed. The boundary conditions were free at both top and bottom; for this reason, the results can be most easily interpreted in terms of a free shear layer.

He first assumed \( l = k \), and studied the variation of \( \sigma \) with \( \text{Re} \), \( \text{Ra} \), and \( k \). The figures below show the variation of \( \sigma \) with \( \text{Re} \) and \( k \) for \( \text{Ra} = 10^5 \). The growth rate generally increases with \( \text{Re} \), for a given scale, and there is a preferred scale for a given \( \text{Re} \). Notice that the growth rate is relatively insensitive to \( \text{Re} \) for \( \text{Re} > 10 \).

The modes whose growth is represented here are of two types. There are long modes which are very sensitive to \( l, k \), and which have a phase speed equal to the mean flow speed at the mid-level. Their growth rates are negative for scales smaller
Fig. 2. Amplification rate of perturbation as a function of the Richardson number $R_i$ (ordinate) and the wavenumber $k^*$ (abscissa). Solid lines are isopleths of amplification rate (in units of 10), dash-dotted line indicates the maximum amplification rate, and dotted line separates the stationary unstable perturbation from the transitive one.

Fig. 3. Variation of amplification rate of perturbation with the wavenumber $k^*$ for different values of $R_i$. Broken lines denote stationary unstable perturbation of higher mode. The numeral labelled at each curve denotes the value of $R_i$. Others are the same as in Fig. 2.

Asai (1970)

than a limiting scale, which depends on $R_i$. Asai calls these "stationary modes."

There are also shorter modes which are relatively insensitive to $k^*$ and which propagate relative to the mid-level flow. Asai calls these "transitive modes." The transitive modes have a maximum possible scale for a given $R_i$. At scales for which both modes occur, the
stationary modes grow faster.

While the stationary modes fill the channel, the transitive modes occupy only the upper or lower portion, depending on their direction of propagation. The figure below shows the phase speeds of the various modes as a function of scale, for different values of \( R_l \). Again, the value of \( R_l \) is 10.5.

---

**Fig. 4.** Phase velocity of unstable perturbation relative to the basic flow at the midlevel as a function of the wavenumber \( k^* \) for different values of \( R_l \).
Growing modes of either type transport temperature upward. The moving modes are confined to either the lower or the upper half of the domain (depending on their direction of propagation). The phases of $\Theta^*$ and $W^*$ agree at the "steering levels" of all modes. The momentum transfer is downward for all modes, so they all gain energy from both the potential and the kinetic energies of the mean flow. The figures below show the relative patterns of vertical velocity and temperature for each type of mode.

Fig. 5. Contours of vertical velocity (solid line) and temperature (broken line) of unstable perturbations in the $\theta-z$ plane for the case of $Ra=10^6$ and $R_l=1$. The ordinate indicates the dimensionless height and the abscissa the phase angle in the $\theta$ direction.

(a) stationary unstable perturbation $(k_x^* = k_y^* = 1)$
(b) transitive unstable perturbation $(k_x^* = k_y^* = 3)$

Asai (1970)
Fig. 6. Vertical profiles of amplitude and phase angle of vertical velocity are shown by thick and thin solid lines, respectively, and those of temperature by thick and thin broken lines respectively. Arrow denotes the steering level. (a) and (b) are the same as in Fig. 5.

Fig. 7. Vertical profiles of vertical transports of heat (solid line) and horizontal momentum $u$ (broken line). (a) and (b) are the same as in Fig. 5.

ASAI (1970)
Asai also studied the sensitivity of the modes to the ratio $\lambda/k$. Let a "transverse mode" be one for which $\lambda/k \ll 1$, and a "longitudinal mode" be one for which $\lambda/k \gg 1$. Asai found that the transverse modes carry momentum upward (against the shear), and so give up energy to the mean flow, while the longitudinal modes carry momentum downward (with the shear), and so gain energy from the mean flow. This important result implies that longitudinal modes will be preferred. Thus shear tends to favor a roll-like planform for convection.
Fig. 8. Variations of vertical momentum transfer $\langle U^*W^* \rangle$ (solid line) and amplification rate of perturbation $\sigma^*$ (broken line) with the ratio between the wavenumbers in the $x$ and $y$ directions, $k_x/k_y$, for different values of $R_1$. The numeral labelled at each curve denotes the value of $R_1$. These are for the case of $R_1 = 10^3$ and $k^* = 2$.

Fig. 9. Contours of the vertical velocity (solid line) and the $x$ component of horizontal velocity in a vertical plane parallel to the basic flow. The warm axis is denoted by a thick dotted line and the cold axis by a thin dotted line: (a) transverse mode, $k_y/k_x = 0.1$, (b) longitudinal mode, $k_y/k_x = 10$. Both are for the case of $R_1 = 1$. Others are the same as in Fig. 8.

Fig. 10. Schematic illustration of a three-dimensional streamline of each perturbation. (a) and (b) are the same as in Fig. 9.

Fig. 11. Energy flow diagram for unstable perturbations. The arrows indicate the direction of the transformations. (a) transverse mode, (b) longitudinal mode.

ASA1 (1970)
5. A hierarchy of transitions

As \( Ra \) is increased to values much above critical, several further "discrete" or sudden transitions in the flow occur, for further "critical" values of \( Ra \). Experiments show that, after each transition, the "efficiency" of heat transport is suddenly increased. The observed regimes are:

1) no motion \[ Ra < (Ra)_{crit} \]

2) steady, two-dimensional flow;

3) steady, three-dimensional flow;

4) time-dependent three-dimensional flow

5) turbulence.

Theoretical understanding of this hierarchy of transitions is not satisfactory, although Malkus (1955) and Busse (1960) have made some progress.
Figure 2. Heat flux vs. Rayleigh number, showing the third and fourth transitions. Heat flux curve, $Pr = 1 \times 10^4$. The heat flux has been non-dimensionalized so that it is the product of Nusselt and Rayleigh numbers.

Figure 3. Heat flux plotted against Rayleigh number, showing hysteresis near the second transition. The Prandtl number is $0.85 \times 10^4$. $\cdots$, $R$ increased; $\circ$, $R$ decreased.
**Figure 1.** Regime diagram. O, steady flows; •, time-dependent flows; *, transition points with observed change in slope; ◻, Rossby's observations of time-dependent flow; □, Willis & Deardorff's (1987a) observations for turbulent flow; △, Silveston's point of transition for time-dependent flow (see text).

**Figure 2.** Heat flux plotted against Rayleigh number, showing the second transition. The heat flux has been non-dimensionalized so that it is the product of Nusselt and Rayleigh numbers. The Prandtl number is $1.0 \times 10^5$. 

\[ R_H=13 R_c \]
E. Conclusion

Each of the instabilities discussed in this chapter is relevant to some aspects of PBL and/or cumulus dynamics. The inflectional instability is the most ubiquitous. In the stable PBL, it is the sole mechanism for maintaining the turbulence. And even in the free convective PBL (in the absence of a mean wind), inflectional instabilities can occur on the boundaries of thermals, where the convective circulations create shear zones. In fact, the latter mechanism is responsible for the chaotic structure of strongly convection, and is essential to the production of turbulence through convection.

A crucially important aspect of these instabilities is that each of them is self-stabilizing, i.e., each of them tends to modify the basic state in such a way as to reduce the growth rate of
Subsequent perturbations of the same type. This has two consequences. First, it means that it is difficult to produce a system which is strongly unstable, since any destabilizing process must fight against the self-stabilizing tendencies of the modes which begin to amplify as soon as the system becomes marginally unstable. Second, it means that the modes which grow in response to the instability will disappear after a period of time unless there is some mechanism which tends to maintain the unstable configuration of the system. These are very general conclusions which we must elaborate and quantify more fully later.
REFERENCES AND BIBLIOGRAPHY

CHAPTER III


